The conic advantage in MINLP

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www.mosek.com

LP NLP

$$\text{LP} \subseteq \text{Conic} \subseteq \text{NLP}$$

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- Lagrangian duality theory can be used to derive conic optimization as the natural extension.

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- Best practices in MINLP leads to conic optimization.

Best practices for MINLP

$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to} & g_i(x) \leq 0, & \forall i = 1, \ldots, k, \\ & x_j \in \mathbb{Z}, & \forall j \in \mathcal{I}. \end{array}$$

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1. Migrate to a linear objective function

From Belotti, Kirches, Leyffer, Linderoth, Luedtke and Mahajan[1]:

$$\min\ x_1 + x_2^2 \quad \longrightarrow \quad \min\ x_1 + t, \quad t \geq x_2^2$$

👍 Avoid interior solutions (which cannot be separated by cuts).

[1] Belotti, Kirches, Leyffer, Linderoth, Luedtke and Mahajan (2013): Mixed-integer nonlinear optimization, Acta Numerica, 22.

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$$\begin{array}{lll} \text{minimize} & c^T x & \\ \text{subject to} & g_i(x) \le 0, & \forall i = 1, \dots, k, \\ & x_j \in \mathbb{Z}, & \forall j \in \mathcal{I}. \end{array}$$

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2. Distinguish convexity

- Avoid unnecessary spatial branching.

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$$\begin{array}{lll} \text{minimize} & c^T x & \\ \text{subject to} & g_i(x) \le 0, & \forall i = 1, \ldots, k, \\ & g_i^{\mathrm{c}}(x) \le 0, & \forall i = 1, \ldots, k^{\mathrm{c}}, \\ & x_j \in \mathbb{Z}, & \forall j \in \mathcal{I}. \end{array}$$

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\end{array}
$$

3. Implement your model using built-in atoms

Lack of callback functions imply:

- Easier to debug the MINLP solver.
- No callback overhead.
- No fear of ill-defined boundary points. For example,
  - The value of $\log(x)$ near $x = 0$.
  - The gradient of $\|x\|_2$ near $x = 0$.

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3. Implement your model using built-in atoms

Possible atoms:

$$t \leq \log(x),$$

(hypograph of logarithm),

$$t \geq \sqrt{x_1^2 + x_2^2},$$

(epigraph of 2-norm).

Use of such built-in atoms imply:

- 👍 Solvers can specialize for stability and high performance.
- 👎 You are limited to a predefined set of atoms.

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\begin{array}{lll}
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To summarize, the set of atoms should be

- Numerically stable,
- Distinguish convexity,
- Versatile (so they can cover a wide range of applications).

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4. Implement solvers to support perspective transformation

- Ceria and Soares[1] characterized the closed convex hull of the union of convex sets using perspective transformation.
  - E.g., to be used on a disjunction $[x \leq 0] \vee [x \geq 1]$.

[1] Ceria and Soares (1999): Convex programming for disjunctive optimization, Mathematical Programming, 86.

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4. Implement solvers to support perspective transformation

- Günlük and Linderoth[2] tightened a bunch of common sets with perspective transformation and promoted it as a useful tool for MINLPs with binary on-off relationships.

[2] Günlük and Linderoth (2012): Perspective reformulation and applications, Mixed Integer Nonlinear Programming, 154.

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4. Implement solvers to support perspective transformation

Support $\tilde{g}(x, s) = g(x/s)$ for $s \ge 0$:

$$t \le \log(x),$$

(hypograph of logarithm),

$$t \ge \sqrt{x_1^2 + x_2^2},$$

(epigraph of 2-norm),

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Support $\tilde{g}(x, s) = g(x/s)$ for $s \geq 0$:

$$t \leq s \log(x/s),$$

(exponential cone),

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Did you notice the abuse of notation?

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4. Implement solvers to support perspective transformation

Hence, rather than:

$$t \le s\, g(x/s),\ s \ge 0$$

we write:

$$t \le s\, g(x/s),\ s > 0$$

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Hence, rather than:

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Journey complete!

1. Migrate to a linear objective function.
2. Distinguish convexity.
3. Implement your model using built-in atoms.
4. Implement solvers to support perspective transformation.

$\Longrightarrow$ **Mixed-integer general conic optimization.**

Resources for mixed-integer conic optimization

- Günlük and Linderoth (2012): Perspective reformulation and applications, Mixed Integer Nonlinear Programming, 154.
- Lodi, Tanneau and Vielma (2020): Disjunctive cuts for Mixed-Integer Conic Optimization.
- Belotti, Goez, Polik, Ralphs and Terlaky (2017): A complete characterization of disjunctive conic cuts for mixed integer second order cone optimization, Discrete Optimization, 24.
- Shahabsafa, Goez and Terlaky (2018): On pathological disjunctions and redundant disjunctive conic cuts. Operations Research Letters, 46(5).
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You are limited to a predefined set of atoms.

Important questions

- What cones to support?
- How to decompose nonlinear constraints into conic atoms?

What cones to support?

- Numerically stable + distinguish convexity + versatile.
- Support perspective transformation

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Idea: **Convex + low degree of nonlinearity (i.e., quadratic-like), so Newton-type methods are expected to solve the relaxations faster.**

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Seminal work by Nesterov and Todd[3]

- Coined a property called "self-scaled".
- Such cones achieve the best known global convergence rate among Newton-type methods.
- Characterized as the set of symmetric cones.

[3] Nesterov and Todd (1997): Self-Scaled Barriers and Interior-Point Methods for Convex Programming, Mathematics of Operations Research, 22.

What cones to support?
The MOSEK selection

3 nonlinear symmetric cones:

- Quadratic: $\{(t,x)\ :\ t \geq \|x\|_2\}$.
- Rotated quadratic: $\{(t,s,x)\ :\ 2ts \geq \|x\|_2^2,\ t \geq 0,\ s \geq 0\}$.
- Semidefinite: $X \succeq 0$.

2 nonsymmetric cones:

- Exponential: $\mathrm{cl}\{(t,s,x)\ :\ t \geq s\exp(x/s),\ s > 0\}$.
- Power: $\{(t,s,x)\ :\ t^{\alpha}s^{1-\alpha} \geq \|x\|_2,\ t \geq 0,\ s \geq 0\}$ for any parameter $0 < \alpha < 1$.

Observation:

- **Almost** all convex optimization problems appearing in practice can be formulated using those 5 cones.

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Observation:

- Lubin et. al.[4]: all convex instances in MINLPLIB 2.0 (i.e., 333) are representable using these cones.

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[4] Lubin, Yamangil, Bent and Vielma (2018): Polyhedral approximation in mixed-integer convex optimization, Mathematical Programming, 172(1).

How to decompose nonlinear constraints?

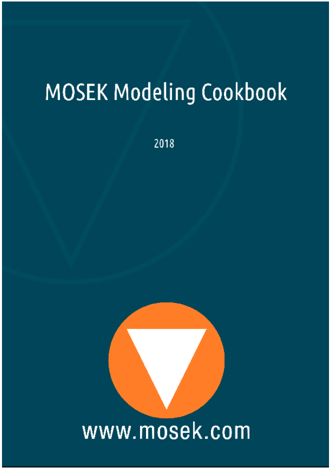
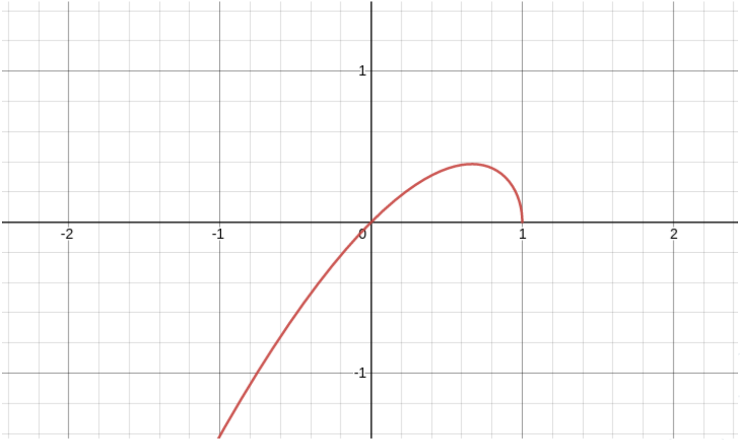


Figure: The MOSEK modeling cookbook (`www.mosek.com/documentation`)

How to decompose nonlinear constraints?

1. $t \leq x\sqrt{1-x}$



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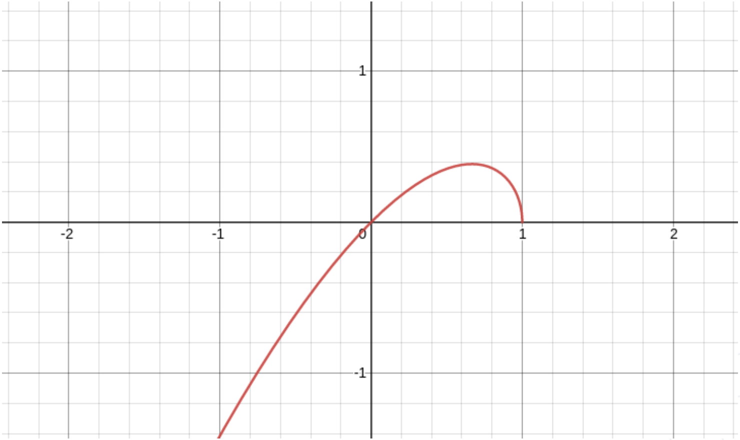
1. $t \leq x\sqrt{1-x} = (x) \cdot (\sqrt{1-x})$.



How to decompose nonlinear constraints?

1. $t \leq x\sqrt{1-x} = (x) \cdot (\sqrt{1-x})$.

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   (Hessian matrix of $y_1 y_2$ is indefinite everywhere).

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3. So relation is important, namely:

$$\left[y_2 = \sqrt{1-y_1}\right] \Leftrightarrow \left[y_1 = 1 - y_2^2\right] \Leftrightarrow \left[y_1 y_2 = y_2 - y_2^3\right].$$

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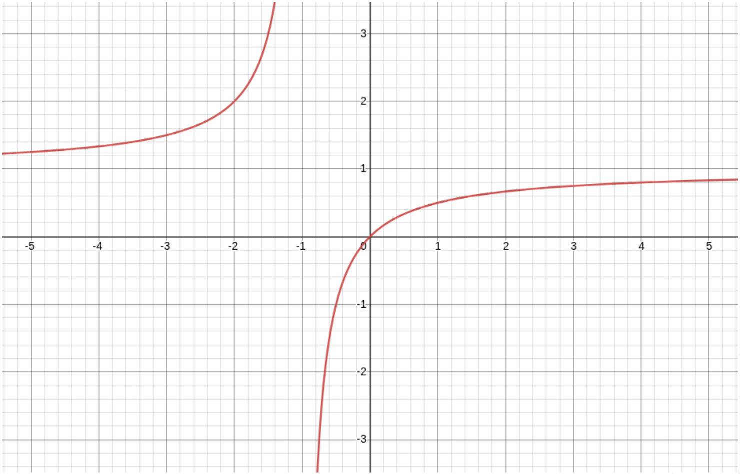
4. $t \leq \sqrt{1-x} - (1-x)^{3/2} = r_1 - r_2$, where

$$r_1 \leq \sqrt{1-x},$$
(rotated quadratic cone),

$$r_2 \geq (1-x)^{3/2},$$
(power cone).

How to decompose nonlinear constraints?

1. $t \le \dfrac{x}{x+1}$.



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How to decompose nonlinear constraints?

1. $t \leq \dfrac{x}{x+1}$.

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Signed sum-of-squares decomposition

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in terms of $u = G^T x + \frac{1}{2} D\tilde{a}$.

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Julia 1.4

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Julia 1.4

How to decompose nonlinear constraints?

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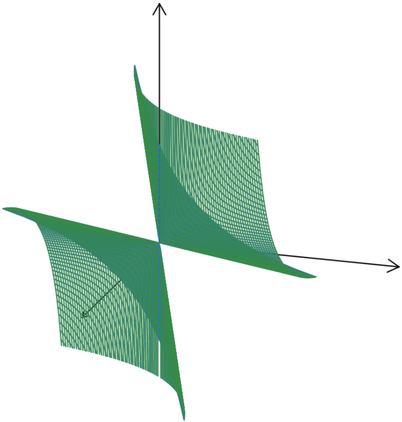
   (Hessian matrix of $xy + yz - zx$ is indefinite everywhere, but we only need a sublevel set!)



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$$\frac{1}{2}(x+y)^2 + \frac{1}{6}(-x+y+2z)^2 \leq \frac{2}{3}(x-y+z)^2,$$

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3. If either $x - y + z \geq 0$ or $x - y + z \leq 0$ we can reformulate using a quadratic cone. Otherwise, you can branch on this disjunction and reformulate in each child node.

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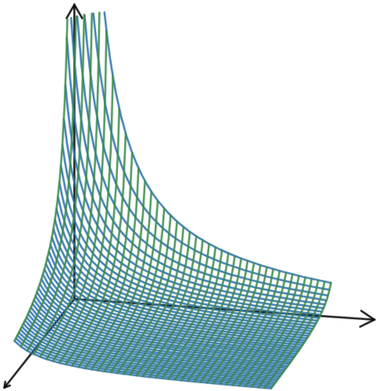

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Summary

- Conic optimization is a sweet-spot in nonlinear optimization.
  - Derivable from LP to preserve nice duality theory.
  - Derivable from MINLP to avoid computational disadvantages.
- No issues with smoothness and differentiability.
- Models can be solved efficiently. Symmetric cones achieve the best known global convergence rate among Newton-type methods.

Further information

- The listed references on mixed-integer conic optimization.
- The MOSEK Modeling Cookbook.
- The MOSEK Optimization Suite (www.mosek.com).
  - Trial and free academic license.
  - Interfaces to C, Java, .NET, Julia, Matlab, R, Python, ...
- The applied optimization tutorials (www.github.com/MOSEK/Tutorials).