Introduction to MOSEK and Conic Optimization

Data-Driven Analytics and Optimization for Energy Systems,
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Optimization

"Classical" continuous problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subj. to} & g_i(x) \leq 0,\ i = 1, \ldots, m, \\ & h_i(x) = 0,\ i = 1, \ldots, k. \end{array}$$

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What about convexity?

$$t \ge x \log(1 + \frac{x}{y}),\ x, y > 0$$

Disciplined Convex Programming (DCP)

New formulation:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax + b \in \mathcal{K} \end{array}$$

where $\mathcal{K}$ is a set which is convex *by construction*.

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Conic optimization:

- $\mathcal{K}$ is a product of *cones*.

Cones in MOSEK

- linear:

$$K = \mathbb{R}_{\geq 0}$$

- quadratic:

$$K = \{x \in \mathbb{R}^n \ : \ x_1 \geq \sqrt{x_2^2 + \cdots + x_n^2}\}$$

- semidefinite:

$$K = \{X \in \mathbb{R}^{n \times n} \ : \ X = FF^T\}$$

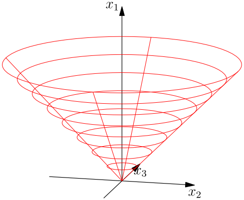
- exponential cone:

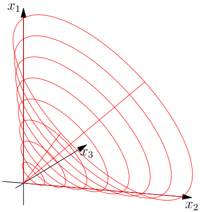
$$K = \{x \in \mathbb{R}^3 \ : \ x_1 \geq x_2 \exp(x_3/x_2),\ x_2 > 0\}$$

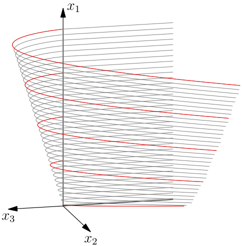
- power cone:

$$K = \{x \in \mathbb{R}^3 \ : \ x_1^{p-1} x_2 \geq |x_3|^p,\ x_1, x_2 \geq 0\},\ p > 1$$

Cones in MOSEK







$x_1 \geq \sqrt{x_2^2 + x_3^2}, \; 2x_1x_2 \geq x_3^2 \qquad x_1 \geq x_2 \exp(x_3/x_2)$

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Conic Modeling Cheatsheet

Cones

Quadratic cone $\mathcal{Q}^n$

$$x_1 \geq \sqrt{x_2^2 + \cdots + x_n^2}$$

Rotated quadratic cone $\mathcal{Q}_r^n$

$$2x_1x_2 \geq x_3^2 + \cdots + x_n^2,\ x_1, x_2 \geq 0$$

Power cone $\mathcal{P}_3^{\alpha,1-\alpha},\ \alpha \in (0,1)$

$$x_1^\alpha x_2^{1-\alpha} \geq |x_3|,\ x_1, x_2 \geq 0$$

Exponential cone $K_{\exp}$

$$x_1 \geq x_2 e^{x_3/x_2},\ x_2 \geq 0$$

Simple bounds

$t \geq x^2$	$(0.5, t, x) \in \mathcal{Q}_r^3$
$\lvert t\rvert \leq \sqrt{x}$	$(0.5, x, t) \in \mathcal{Q}_r^3$
$t \geq \lvert x\rvert$	$(t, x) \in \mathcal{Q}^2$
$t \geq 1/x,\ x > 0$	$(x, t, \sqrt{2}) \in \mathcal{Q}_r^3$
$t \geq \lvert x\rvert^p,\ p > 1$	$(t, 1, x) \in \mathcal{P}_3^{1/p, 1-1/p}$
$t \geq 1/x^p,\ x > 0,\ p > 0$	$(t, x, 1) \in \mathcal{P}_3^{1/(1+p), p/(1+p)}$
$\lvert t\rvert \leq x^p,\ x > 0,\ p \in (0,1)$	$(x, 1, t) \in \mathcal{P}_3^{p, 1-p}$
$t \geq \lvert x\rvert^p / y^{p-1},\ y \geq 0$ $p > 1$	$(t, y, x) \in \mathcal{P}_3^{1/p, 1-1/p}$
$t \geq x^T x / y,\ y \geq 0$	$(0.5t, y, x) \in \mathcal{Q}_r^{n+2}$
$t \geq e^x$	$(t, 1, x) \in K_{\exp}$
$t \leq \log x$	$(x, 1, t) \in K_{\exp}$
$t \geq 1/\log x,\ x > 1$	$(u, t, \sqrt{2}) \in \mathcal{Q}_r^3$ $(x, 1, u) \in K_{\exp}$
$t \geq a_1^{x_1} \cdots a_n^{x_n},\ a_i > 0$	$(t, 1, \sum x_i \log a_i) \in K_{\exp}$
$t \geq x e^x,\ x \geq 0$	$(t, x, u) \in K_{\exp}$ $(0.5, u, x) \in \mathcal{Q}_r^3$
$t \geq \log(1 + e^x)$	$u + v \leq 1$ $(u, 1, x - t) \in K_{\exp}$ $(v, 1, -t) \in K_{\exp}$
$t \geq \lvert x\rvert^{3/2}$	$(t, 1, x) \in \mathcal{P}_3^{2/3, 1/3}$
$t \geq x^{3/2},\ x \geq 0$	$(s, t, x), (x, 1/8, s) \in \mathcal{Q}_r^3$
$t \geq 1/x^3,\ x > 0$	$(t, x, 1) \in \mathcal{P}_3^{3/4, 1/4}$
$0 \leq t \leq x^{2/5},\ x \geq 0$	$(x, 1, t) \in \mathcal{P}_3^{2/5, 3/5},\ t \geq 0$

Means and averaging

Log-sum-exp $t \geq \log(\sum e^{x_i})$	$(z_i, 1, x_i - t) \in K_{\exp}$ $i = 1, \ldots, n$ $\sum z_i \leq 1$
Harmonic mean $0 \leq t \leq n(\sum x_i^{-1})^{-1}$ $x_i > 0$	$(z_i, x_i, t) \in \mathcal{Q}_r^3$ $i = 1, \ldots, n$ $\sum z_i = nt/2$
Geometric mean $\lvert t\rvert \leq (x_1 \cdots x_n)^{1/n}$ $x_i > 0$	$(z_i, x_i, z_{i+1}) \in \mathcal{P}_3^{1-1/i, 1/i}$ $i = 2, \ldots, n$ $z_2 = x_1,\ z_{n+1} = t$
$\lvert t\rvert \leq \sqrt{xy},\ x, y > 0$	$(x, y, \sqrt{2}t) \in \mathcal{Q}_r^3$
Weighted geom. mean $\lvert t\rvert \leq x_1^{\alpha_1} \cdots x_n^{\alpha_n},\ x_i > 0$ $\alpha_i > 0, \sum \alpha_i = 1$	$(z_i, x_i, z_{i+1}) \in \mathcal{P}_3^{1-\beta_i, \beta_i}$ $\beta_i = \alpha_i / (\alpha_1 + \cdots + \alpha_i)$ $i = 2, \ldots, n$ $z_2 = x_1,\ z_{n+1} = t$
$\lvert t\rvert \leq x^{1/4} y^{5/12} z^{1/3}$ $x, y, z \geq 0$	$(s, z, t) \in \mathcal{P}_3^{2/3, 1/3}$ $(x, y, s) \in \mathcal{P}_3^{3/8, 5/8}$

Entropy

$t \leq -x \log x$	$(1, x, t) \in K_{\exp}$
$t \geq x \log(x/y)$	$(y, x, -t) \in K_{\exp}$
$t \geq \log(1 + 1/x)$ $x > 0$	$(x + 1, u, \sqrt{2}) \in \mathcal{Q}_r^3$ $(1 - u, 1, -t) \in K_{\exp}$
$t \leq \log(1 - 1/x)$ $x > 1$	$(x, u, \sqrt{2}) \in \mathcal{Q}_r^3$ $(1 - u, 1, t) \in K_{\exp}$
$t \geq x \log(1 + x/y)$ $x, y > 0$	$(y, x + y, u) \in K_{\exp}$ $(x + y, y, v) \in K_{\exp}$ $t + u + v = 0$

Convex quadratic problems

Let $\Sigma \in \mathbb{R}^{n \times n}$, symmetric, p.s.d.
Find $\Sigma = LL^T$, $L \in \mathbb{R}^{n \times k}$ (Cholesky factor).
Then $x^T \Sigma x = \lVert L^T x\rVert_2^2$.

$t \geq \frac{1}{2} x^T \Sigma x$	$(1, t, L^T x) \in \mathcal{Q}_r^{k+2}$
$t \geq \sqrt{x^T \Sigma x}$	$(t, L^T x) \in \mathcal{Q}^{k+1}$
$\frac{1}{2} x^T \Sigma x + p^T x + q \leq 0$	$(1, -p^T x - q, L^T x) \in \mathcal{Q}_r^{k+2}$
$\max_x c^T x - \frac{1}{2} x^T \Sigma x$	$\max c^T x - r$ $(1, r, L^T x) \in \mathcal{Q}_r^{k+2}$
$c^T x + d \geq \lVert Ax + b\rVert_2$	$(c^T x + d, Ax + b) \in \mathcal{Q}^{m+1}$

Norms, $x \in \mathbb{R}^n$

$\lVert \cdot \rVert_1,\ t \geq \sum \lvert x_i\rvert$	$(z_i, x_i) \in \mathcal{Q}^2,\ t = \sum z_i$
$\lVert \cdot \rVert_2,\ t \geq (\sum x_i^2)^{1/2}$	$(t, x) \in \mathcal{Q}^{n+1}$
$\lVert \cdot \rVert_p,\ p > 1$ $t \geq (\sum \lvert x_i\rvert^p)^{1/p}$	$(z_i, t, x_i) \in \mathcal{P}_3^{1/p, 1-1/p}$ $i = 1, \ldots, n$ $\sum z_i = t$

Geometry

Bounding ball $\min_x \max_i \lVert x - x_i\rVert_2$	$\min r$ $(r, x - x_i) \in \mathcal{Q}^{n+1}$
Geometric median $\min_x \sum \lVert x - x_i\rVert_2$	$\min \sum t_i$ $(t_i, x - x_i) \in \mathcal{Q}^{n+1}$
Analytic center $\max_x \sum \log(b_i - a_i^T x)$	$\max \sum t_i$ $(b_i - a_i^T x, 1, t_i) \in K_{\exp}$

Regression and fitting

Regularized least squares $\min_w \lVert Xw - y\rVert_2^2 + \lambda \lVert w\rVert_2^2$	$\min t + \lambda r$ $(0.5, t, Xw - y) \in \mathcal{Q}_r^{m+2}$ $(0.5, r, w) \in \mathcal{Q}_r^{n+2}$
Max likelihood $\max_p p_1^{a_1} \cdots p_n^{a_n}$	$\max \sum a_i t_i$ $(p_i, 1, t_i) \in K_{\exp}$
Logistic cost function $t \geq -\log(1/(1 + e^{-\theta^T x}))$	$u + v \leq 1$ $(u, 1, -\theta^T x - t) \in K_{\exp}$ $(v, 1, -t) \in K_{\exp}$

Risk-return

$\Sigma \in \mathbb{R}^{n \times n}$ – covariance, $\Sigma = LL^T,\ L \in \mathbb{R}^{n \times k}$

$\max_x \alpha^T x$ s.t. $x^T \Sigma x \leq \gamma$	$\max_x \alpha^T x$ $(\sqrt{\gamma}, L^T x) \in \mathcal{Q}^{k+1}$
$\max_x \alpha^T x - \delta x^T \Sigma x$	$\max_x \alpha^T x - \delta r$ $(0.5, r, L^T x) \in \mathcal{Q}^{k+2}$
Risk plus $x^{1.5}$ impact cost $t \geq \delta x^T \Sigma x + \beta \sum \lvert x_i\rvert^{3/2}$	$t \geq \delta r + \beta \sum u_i$ $(0.5, r, L^T x) \in \mathcal{Q}^{k+2}$ $(u_i, 1, x_i) \in \mathcal{P}_3^{2/3, 1/3}$
Risk in factor model $\gamma \geq x^T (D + FSF^T) x$ D – specific risk (diag.) $F \in \mathbb{R}^{n \times k}$ – factor loads $S = UU^T$ – factor cov.	$\gamma \geq t + s$ $(0.5, t, \sqrt{D} x) \in \mathcal{Q}_r^{n+2}$ $(0.5, s, U^T F^T x) \in \mathcal{Q}_r^{k+2}$

Challenge

If you happen to know a

- natural,
- practical,
- important,
- convex

optimization problem, which cannot be expressed using the cones we have so far, let us know!

Conic ecosystem

Modeling languages

- CVX (Matlab) `http://ask.cvxr.com`
- CVXPY (Python) `http://www.cvxpy.org/`
- YALMIP (Matlab) `https://yalmip.github.io/`
- JuMP (Julia) `https://github.com/JuliaOpt/JuMP.jl`
- $\cdots$

Free solvers

- ECOS, SCS, SDPT3, Sedumi, $\cdots$

Conic ecosystem: Mosek Fusion

$$\begin{aligned} \min \quad & \sum c_i P_{G_i} \\ \text{s.t.} \quad & P_{G_i}^{min} \leq P_{G_i} \leq P_{G_i}^{max} \\ & \mathbf{B} \cdot \theta = \mathbf{P_G} - \mathbf{P_D} \\ & (\theta_i - \theta_j)/x_{ij} \leq P_{ij,max} \end{aligned}$$

```
def dcopf(ng, nb, PGmin, PGmax, Pmax, B, x, PD, c):
  M = Model()
  PG = M.variable(ng, Domain.inRange(PGmin, PGmax))
  theta = M.variable(nb)

  M.constraint(Expr.sub(Expr.mul(B, theta), Expr.sub(PG, PD)),
               Domain.equalsTo(0.0))
  M.constraint(Expr.sub(Var.hrepeat(theta,nb),
                        Var.vrepeat(theta.transpose(),nb)),
               Domain.lessThan(Pmax*x))

  M.objective(ObjectiveSense.Minimize, Expr.dot(c, PG))
```

Optimizers in MOSEK

- primal/dual simplex, interior-point, MIP

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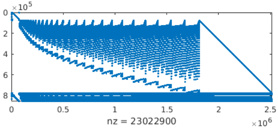

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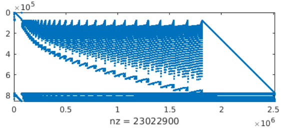

- LP, QP, SOCP, SDP, exponential, power cone
- Low-level optimization API
  - C, Python, Java, .NET, Matlab, R, Julia
- Object-oriented API *Fusion*
  - C++, Python, Java, .NET
- 3rd party
  - GAMS, AMPL, CVX, CVXOPT, CVXPY, YALMIP, Pyomo, GPkit, JuMP

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Thank you!