Extending MOSEK with exponential cones

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Conic optimization

Linear cone problem:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \in K, \end{array}$$

with $K = K_1 \times K_2 \times \cdots \times K_p$ a product of proper cones.

Dual:

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & c - A^T y = s \\ & s \in K^*, \end{array}$$

with $K^* = K_1^* \times K_2^* \times \cdots \times K_p^*$.

Symmetric cones (supported by MOSEK 8)

- *the nonnegative orthant*

$$K_l^n := \{x \in \mathbb{R}^n \mid x_j \geq 0,\ j = 1, \ldots, n\},$$

- *the quadratic cone*

$$K_q^n = \{x \in \mathbb{R}^n \mid x_1 \geq \left(x_2^2 + \cdots + x_n^2\right)^{1/2}\},$$

- *the rotated quadratic cone*

$$K_r^n = \{x \in \mathbb{R}^n \mid 2x_1x_2 \geq x_3^2 + \ldots x_n^2,\ x_1, x_2 \geq 0\}.$$

- *the semidefinite matrix cone*

$$K_s^n = \{x \in \mathbb{R}^{n(n+1)/2} \mid z^T \mathbf{mat}(x) z \geq 0,\ \forall z\}.$$

Nonsymmetric cones (supported by MOSEK 9)

- *the three-dimensional power cone*

$$K_{\text{pow}}^{\alpha} = \{x \in \mathbb{R}^3 \mid x_1^{\alpha} x_2^{(1-\alpha)} \geq |x_3|,\ x_1, x_2 > 0\},$$

  for $0 < \alpha < 1$.

- *the exponential cone*

$$K_{\text{exp}} = \mathrm{cl}\{x \in \mathbb{R}^3 \mid x_1 \geq x_2 \exp(x_3/x_2),\ x_2 > 0\}.$$

Central path for conic problem

Central path for homogenous model parametrized by μ:

$$
\begin{aligned}
Ax_\mu - b\tau_\mu &= \mu(Ax - b\tau)\\
s_\mu + A^T y_\mu - c\tau_\mu &= \mu(s + A^T y - c\tau)\\
c^T x_\mu - b^T y_\mu + \kappa_\mu &= \mu(c^T x - b^T y + \kappa)
\end{aligned}
$$

$$
s_\mu = -\mu F'(x_\mu), \quad x_\mu = -\mu F_*'(s_\mu), \quad \kappa_\mu \tau_\mu = \mu,
$$

or equivalently

$$
\begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix}
\begin{bmatrix} y_\mu \\ x_\mu \\ \tau_\mu \end{bmatrix}
-
\begin{bmatrix} 0 \\ s_\mu \\ \kappa_\mu \end{bmatrix}
= \mu
\begin{bmatrix} r_p \\ r_d \\ r_g \end{bmatrix}
$$

$$
s_\mu = -\mu F'(x_\mu), \quad x_\mu = -\mu F_*'(s_\mu), \quad \kappa_\mu \tau_\mu = \mu,
$$

$$
r_p := Ax - b\tau, \quad r_d := c\tau - A^T y - s, \quad r_g := \kappa - c^T x + b^T y, \quad r_c := x^T s + \tau\kappa.
$$

Scaling for nonsymmetric cones

Following Tunçel [3] we consider a scaling $W^T W \succ 0$,

$$v = Wx = W^{-T} s, \qquad \tilde{v} = W\tilde{x} = W^{-T}\tilde{s}$$

where $\tilde{x} := -F'_*(s)$ and $\tilde{s} := -F'(x)$. The centrality conditions

$$x = \mu\tilde{x}, \quad s = \mu\tilde{s}$$

can then be written symmetrically as

$$v = \mu\tilde{v},$$

and we linearize the centrality condition $v = \mu\tilde{v}$ as

$$W\Delta x + W^{-T}\Delta s = -v + \mu\tilde{v}.$$

A centering search-direction

Let $\mu := \dfrac{x^T s + \tau\kappa}{\nu + 1}$ with barrier parameter ν and centering γ.

$$\begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix} \begin{bmatrix} \Delta y_c \\ \Delta x_c \\ \Delta \tau_c \end{bmatrix} - \begin{bmatrix} 0 \\ \Delta s_c \\ \Delta \kappa_c \end{bmatrix} = (\gamma - 1) \begin{bmatrix} r_p \\ r_d \\ r_g \end{bmatrix}$$

$$W \Delta x_c + W^{-T} \Delta s_c = \gamma \mu \tilde{v} - v, \quad \tau \Delta \kappa_c + \kappa \Delta \tau_c = \gamma \mu - \kappa \tau,$$

Constant decrease of residuals and complementarity:

$$\begin{aligned} Ax^+ - b\tau^+ &= \eta \cdot r_p, \\ c\tau^+ - A^T y^+ - s^+ &= \eta \cdot r_d, \\ b^T y^+ - c^T x^+ - \kappa^+ &= \eta \cdot r_g, \\ (x^+)^T s^+ + \tau^+ \kappa^+ &= \eta \cdot r_c, \end{aligned}$$

where $z^+ := (z + \alpha \Delta z_c)$ and $\eta = (1 - \alpha(1 - \gamma))$.

A higher-order corrector term

Derivatives of $s_\mu = -\mu F'(x_\mu)$:

$$\begin{aligned} \dot{s}_\mu + \mu F''(x_\mu)\dot{x}_\mu &= -F'(x_\mu), \\ \ddot{s}_\mu + \mu F''(x_\mu)\ddot{x}_\mu &= -2F''(x_\mu)\dot{x}_\mu - \mu F'''(x_\mu)[\dot{x}_\mu, \dot{x}_\mu]. \end{aligned}$$

Since

$$\mu \dot{x}_\mu = -[F''(x_\mu)]^{-1}(F'(x_\mu) + \dot{s}_\mu) = x_\mu - [F''(x_\mu)]^{-1}\dot{s}_\mu,$$

we have

$$\mu F'''(x_\mu)[\dot{x}_\mu, \dot{x}_\mu] = \underbrace{F'''(x_\mu)[\dot{x}_\mu, x_\mu]}_{-2F''(x_\mu)\dot{x}_\mu} - F'''(x_\mu)[\dot{x}_\mu, (F''(x_\mu))^{-1}\dot{s}_\mu]$$

so

$$\ddot{s}_\mu + \mu F''(x_\mu)\ddot{x}_\mu = F'''(x_\mu)[\dot{x}_\mu, (F''(x_\mu))^{-1}\dot{s}_\mu].$$

An affine search-direction

Affine search-direction:

$$\begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix} \begin{bmatrix} \Delta y_a \\ \Delta x_a \\ \Delta \tau_a \end{bmatrix} - \begin{bmatrix} 0 \\ \Delta s_a \\ \Delta \kappa_a \end{bmatrix} = - \begin{bmatrix} r_p \\ r_d \\ r_g \end{bmatrix}$$

$$W \Delta x_a + W^{-T} \Delta s_a = -v, \qquad \tau \Delta \kappa_a + \kappa \Delta \tau_a = -\kappa \tau,$$

satisfies

$$(\Delta x_a)^T \Delta s_a + \Delta \tau_a \Delta \kappa_a = 0.$$

Since

$$\dot{s}_\mu + \mu F''(x_\mu) \dot{x}_\mu = -F'(x_\mu) = s_\mu,$$

we interpret $\Delta s_a = -\dot{s}_\mu$ and $\Delta x_a = -\dot{x}_\mu$.

A higher-order corrector term

From

$$\ddot{s}_\mu + \mu F''(x_\mu)\ddot{x}_\mu = F'''(x_\mu)[\dot{x}_\mu, (F''(x_\mu))^{-1}\dot{s}_\mu]$$

we define a corrector direction as

$$W\Delta x_{\mathrm{cor}} + W^{-T}\Delta s_{\mathrm{cor}} = \frac{1}{2}W^{-T}F'''(x)[\Delta x_{\mathrm{a}}, (F''(x))^{-1}\Delta s_{\mathrm{a}}].$$

Note that

$$\begin{aligned} s^T\Delta x_{\mathrm{cor}} + x^T\Delta s_{\mathrm{cor}} &= \frac{1}{2}x^T F'''(x)[\Delta x_{\mathrm{a}}, (F''(x))^{-1}\Delta s_{\mathrm{a}}] \\ &= -(\Delta x_{\mathrm{a}})^T\Delta s_{\mathrm{a}}, \end{aligned}$$

condition for constant decrease of complementarity.

A higher-order corrector term

- Linear case,

$$\frac{1}{2}F'''(x)[\Delta x_a, (F''(x))^{-1}\Delta s_a] = -\mathbf{diag}(x)^{-1}\mathbf{diag}(\Delta x_a)\Delta s_a,$$

- Semidefinite case,

$$\begin{aligned}\frac{1}{2}F'''(x)[\Delta x_a, (F''(x))^{-1}\Delta s_a] &= -\frac{1}{2}x^{-1}\Delta x_a \Delta s_a - \frac{1}{2}\Delta s_a \Delta x_a x^{-1} \\ &= -(x^{-1}) \circ (\Delta x_a \Delta s_a),\end{aligned}$$

- Second-order cone case,

$$F'''(x)[(F''(x))^{-1}u] = -\frac{2}{x^T Q x}(u x^T Q + Q x u^T - (x^T u) Q)$$

for $Q = \mathbf{diag}(1, -1, \ldots, -1)$. Then

$$F'''(x)[(F''(x))^{-1}u]e = -2(x^{-1} \circ u).$$

Combined centering-corrector direction

A combined centering-corrector direction:

$$\begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta \tau \end{bmatrix} - \begin{bmatrix} 0 \\ \Delta s \\ \Delta \kappa \end{bmatrix} = (\gamma - 1) \begin{bmatrix} r_p \\ r_d \\ r_g \end{bmatrix}$$

$$W \Delta x + W^{-1} \Delta s = \gamma \mu \tilde{v} - v + \frac{1}{2} W^{-T} F'''(x)[\Delta x_{\mathrm{a}}, (F''(x))^{-1} \Delta s_{\mathrm{a}}],$$

$$\tau \Delta \kappa + \kappa \Delta \tau = \gamma \mu - \tau \kappa - \Delta \tau_{\mathrm{a}} \Delta \kappa_{\mathrm{a}}.$$

All residuals and complementarity decrease by η.

Computing the scaling matrix

Theorem (Schnabel [2])

Let $S, Y \in \mathbb{R}^{n\times p}$ have full rank p. Then there exists $H \succ 0$ such that $HS = Y$ if and only if $Y^T S \succ 0$.

As a consequence

$$H = Y(Y^T S)^{-1} Y^T + ZZ^T$$

where $S^T Z = 0$, $\mathbf{rank}(Z) = n - p$. We have $n = 3$, $p = 2$ and

$$S := \begin{pmatrix} x & \tilde{x} \end{pmatrix}, \quad Y := \begin{pmatrix} s & \tilde{s} \end{pmatrix},$$

with

$$\det(Y^T S) = \nu^2(\mu\tilde{\mu} - 1) \geq 0$$

vanishing only on the central path.

Computing the scaling matrix

Any scaling with $n = 3$ satisfies

$$W^T W = Y(Y^T S)^{-1} Y^T + zz^T$$

where $\begin{pmatrix} x & \tilde{x} \end{pmatrix}^T z = 0$, $z \neq 0$. Expanding the BFGS update [2]

$$H^+ = H + Y(Y^T S)^{-1} Y^T - HS(S^T HS)^{-1} S^T H,$$

for $H \succ 0$ gives the scaling by Tunçel [3] and Myklebust [1], *i.e.*,

$$zz^T = H - HS(S^T HS)^{-1} S^T H,$$

with $H = \mu F''(x)$.

A negative result on complexity

Nesterov's long-step Hessian estimation property holds if

$$F'''(x)[u] \preceq 0, \quad \forall x \in \mathbf{int}(K),\ \forall u \in K.$$

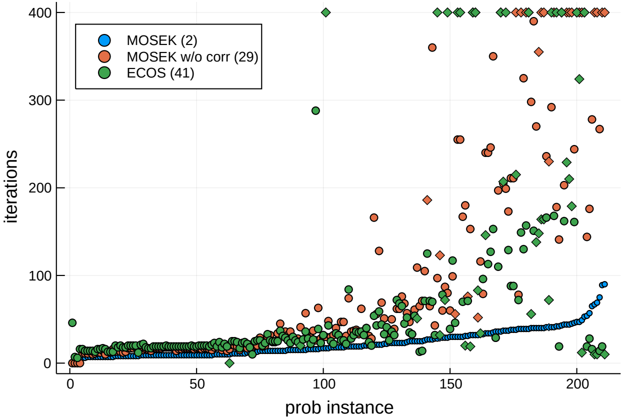
We have

$$F'''([1;1;-1])[u] =$$

$$u_1 \begin{bmatrix} -9 & 6 & 3 \\ 6 & -5 & -3 \\ 3 & -3 & -2 \end{bmatrix} + u_2 \begin{bmatrix} 6 & -5 & -3 \\ -5 & 2 & 3 \\ -3 & 3 & 2 \end{bmatrix} + u_3 \begin{bmatrix} 3 & -3 & -2 \\ -3 & -3 & 2 \\ -2 & 2 & 2 \end{bmatrix}.$$

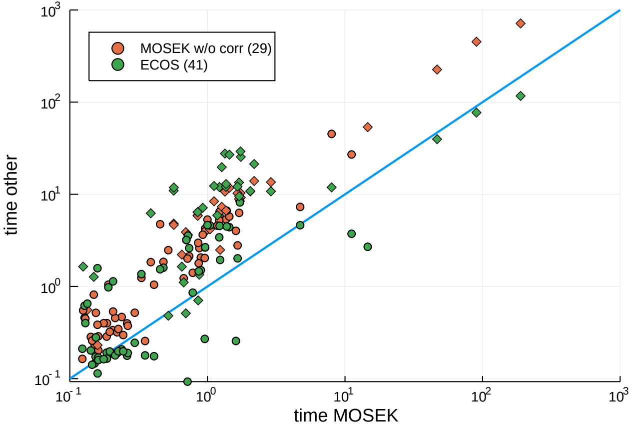
Not negative semidefinite for all $u \in K$.

Comparing MOSEK and ECOS conic solvers



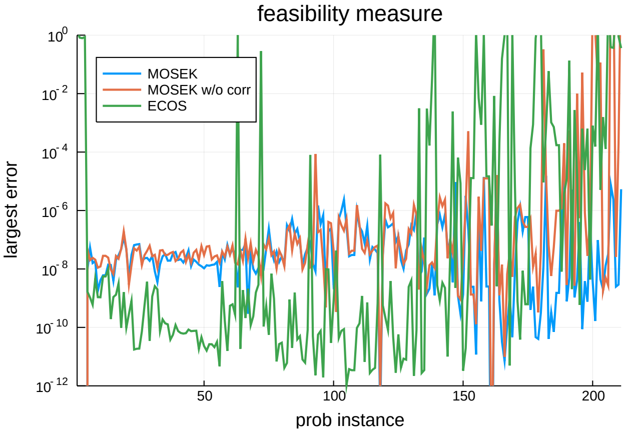
Iteration counts for different exponential cone problems. Failures marked with $\diamond$.

Comparing MOSEK and ECOS conic solvers



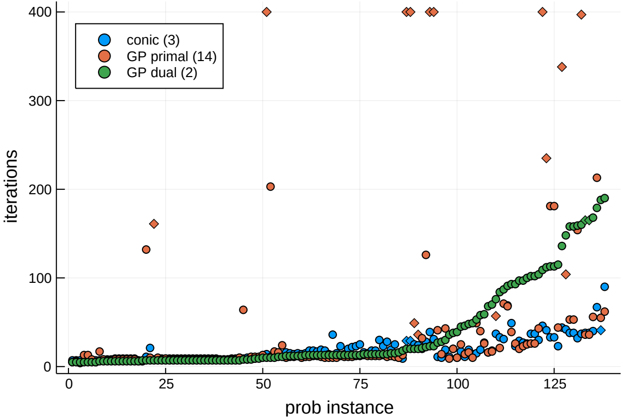
Solution time for different exponential cone problems. Failures marked with $\diamond$.

Comparing MOSEK and ECOS conic solvers



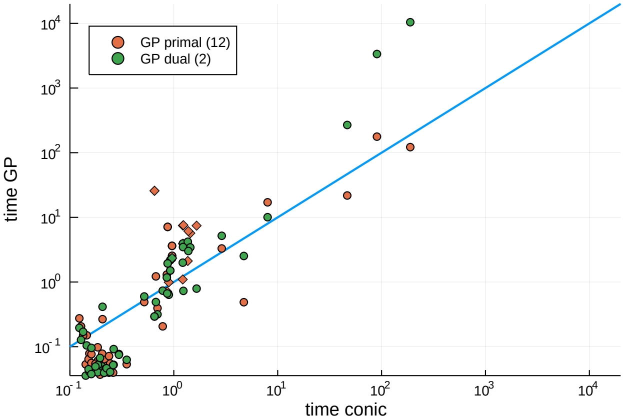
Feasibility measures for different exponential cone problems.

Comparing MOSEK conic and MOSEK GP solvers



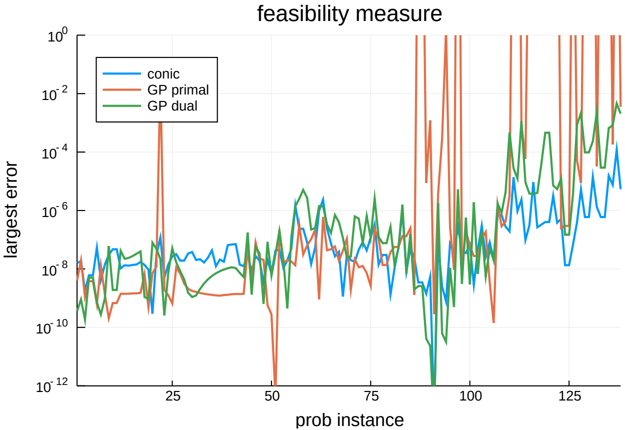
Iteration counts for different GPs. Failures marked with $\diamond$.

Comparing MOSEK conic and MOSEK GP solvers



Solution time for different GPs. Failures marked with $\diamond$.

Comparing MOSEK conic and MOSEK GP solvers



Feasibility measures for different GPs.

References

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[3] L. Tunçel.
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