Exponential cone in MOSEK

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Michał Adamaszek, MOSEK ApS

www.mosek.com

MOSEK

- linear conic solver: SOCP, SDP, EXP, POW,
- primal/dual simplex for LPs,
- convex QPs,
- + mixed-integer,
- APIs: MATLAB, C, Python, Java, .NET, R, Julia,
- conic modeling language *Fusion*, C++, Java, .NET, Python,
- third party: AMPL, GAMS, CVX, CVXPY, YALMIP, JuMP
- version 9 (soon).

Conic problems

A conic problem in canonical primal form:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{s.t.} & Ax = b \\ & x \in \mathcal{K} \end{array}$$

with dual

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{s.t.} & c - A^T y \in \mathcal{K}^* \end{array}$$

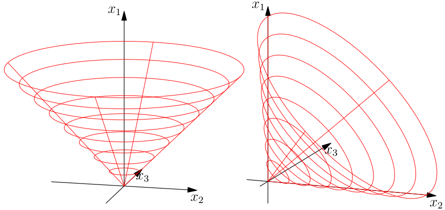
where $\mathcal{K} = K_1 \times \cdots \times K_s$ is a product of cones.

Extremely disciplined convex programming: a problem in conic form is convex by construction.

Conic problems

Nonlinear symmetric cones supported in MOSEK:

- quadratic (SOC) and rotated quadratic:



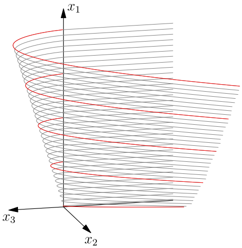
$$x_1 \geq (x_2^2 + \cdots + x_n^2)^{1/2}, \quad 2x_1x_2 \geq x_3^2 + \cdots + x_n^2$$

- semidefinite:

$$\mathbb{S}_+^n = \{X \in \mathbb{R}^{n\times n} \ : \ X = FF^T\}$$

Exponential cone

$$K_{\exp} = \text{cl}\ \{x \in \mathbb{R}^3\ :\ x_1 \geq x_2 \exp(x_3/x_2),\ x_1, x_2 > 0\}$$



Equivalently

$$-x_3 \geq x_2 \log x_2/x_1 = \texttt{rel\_entr}(x_2, x_1)$$

or the perspective cone (epigraph of the perspective function $(x, y) \to x f(y/x)$) for either $f(u) = \exp(u)$ or $f(u) = u \log(u)$.

Modeling with the exponential cone

- $t \geq \exp(x) \iff (t, 1, x) \in K_{\exp}$
- $t \leq \log(x) \iff (x, 1, t) \in K_{\exp}$
- $t \geq a_1^{x_1} \cdots a_k^{x_k} \iff (t, 1, \sum x_i \log a_i) \in K_{\exp},\ \ a_i \in \mathbb{R}_+$
- $t \geq x \exp(x)$

$$\begin{array}{ll} t \geq x \exp(y/x) & (t, x, y) \in K_{\exp} \\ y \geq x^2 & (0.5, y, x) \in \mathcal{Q}_r \end{array}$$

Modeling with the exponential cone

What is (SOC,EXP,POW,SDP) — representable? Probably a lot.

From ask.cvxr.com:

Hi guys,
I met a problem while using cvx. My objective function is f(x,y) = x*log(1+x/y),x>0,y>0. The function is convex, but I don't know how to express which in cvx. Can you help me?

I believe `x*log(1+x/y)` is convex for x > 0, y > 0.

I don't see how to get it into CVX. Perhaps someone else knows how, although I don't know whether it can be done.

```
x*log(1+x/y)=rel_entr(x+y,y)+rel_entr(y,x+y)
```

Modeling with the exponential cone

- Product of variables in the objective

$$\max(x_1 x_2 \cdots x_n) \iff \max(\sum \log x_i)$$

  Appears in maximum likelihood optimization.

- Log-sum-exp

$$t \geq \log(e^{x_1} + \cdots + e^{x_n})$$

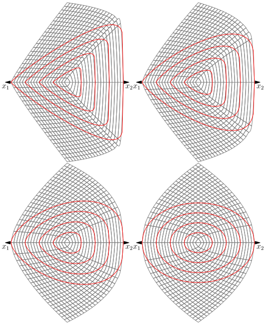
  is equivalent to

$$e^{x_1 - t} + \cdots + e^{x_n - t} \leq 1.$$

Power cone

$$K_{\text{pow}}^{p} = \{x \in \mathbb{R}^3 \;:\; x_1^{p-1} x_2 \geq |x_3|^p,\; x_1, x_2 > 0\},\; p > 1$$

- generalizes the Lorentz cone ($p = 2$),
- is also a perspective cone (of $f(u) = |u|^p$),
- allows modeling of x^p, $\|x\|_p$, etc.



Geometric programming

A geometric program (GP) has the form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{s.t.} & f_j(x) \leq 1, \;\; j = 1, \ldots, m \\ & x_i > 0, \;\; i = 1, \ldots, n. \end{array}$$

where each f is a *posynomial*:

$$f(x) = \sum_j c_k x^{\alpha_k}, \;\; c_k > 0, \alpha_k \in \mathbb{R}^n,$$

e.g. $2\sqrt{x} + 0.1x^{-1}z^3 \leq 1.$
For $x_i = \exp(y_i)$ constraints take a convex (conic) form

$$\sum_k c_k \exp(\alpha_k^T y_k) \leq 1.$$

Applications: circuit design, chemical engineering, mechanical engineering, wireless networks, ...

Logistic regression

Training data: $(x_1, y_1), \dots, (x_n, y_n) \in \mathbb{R}^d \times \{0, 1\}$.
Classify new data using

$$h_\theta(x) = \frac{1}{1 + \exp(-\theta^T x)} \sim P[y = 1].$$

Cost function

$$J(\theta) = \sum_i -y_i \log(h_\theta(x_i)) - (1 - y_i) \log(1 - h_\theta(x_i)).$$

Regularized optimization problem

$$\text{minimize}_{\theta \in \mathbb{R}^d} \; J(\theta) + \lambda \|\theta\|_2.$$

Logistic regression — conic model

$$\text{minimize}_{\theta\in\mathbb{R}^d} \sum_i -y_i \log(h_\theta(x_i)) - (1-y_i)\log(1-h_\theta(x_i)) + \lambda\|\theta\|_2.$$

Formulate as:

$$\begin{array}{lllr}
\text{minimize} & \mathbf{1}^T t_i + \lambda r & & \\
\text{s.t} & t_i \geq -\log(h_\theta(x)) & = \log(1+\exp(-\theta^T x_i)) & \text{if } y_i = 1, \\
 & t_i \geq -\log(1-h_\theta(x)) & = \log(1+\exp(\theta^T x_i)) & \text{if } y_i = 0, \\
 & r \geq \|\theta\|_2, & &
\end{array}$$

Each constraint is conic-representable:

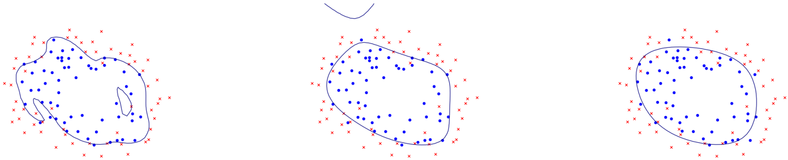
- $r \geq \|\theta\|_2 \iff (r,\theta) \in \mathcal{Q}$
- $t \geq \log(1+\exp(u)) \iff \exp(-t) + \exp(u-t) \leq 1 \iff$

$$\begin{array}{rcl}
y_1 + y_2 & \leq & 1, \\
(y_1, 1, u-t) & \in & K_{\exp}, \\
(y_2, 1, -t) & \in & K_{\exp}.
\end{array}$$

Logistic regression in Fusion

```python
# t >= log( 1 + exp(u) )
def softplus(M, t, u):
    y = M.variable(2)
    # y_1 + y_2 <= 1
    M.constraint(Expr.sum(y), Domain.lessThan(1.0))
    # [ y_1  1  u-t ]
    # [ y_2  1  -t  ]  in ExpCone
    M.constraint(Expr.hstack(y, Expr.constTerm(2, 1.0),
                                Expr.vstack(Expr.sub(u,t), Expr.neg(t))),
                 Domain.inPExpCone())

def logisticRegression(X, y, lamb=1.0):
    n, d = X.shape         # num samples, dimension
    M = Model()
    theta = M.variable(d)
    t     = M.variable(n)
    reg   = M.variable()

    M.objective(ObjectiveSense.Minimize, Expr.add(Expr.sum(t), Expr.mul(lamb,reg)))
    M.constraint(Var.vstack(reg, theta), Domain.inQCone())

    for i in range(n):
        dot = Expr.dot(X[i], theta)
        if y[i]==1: softplus(M, t.index(i), Expr.neg(dot))
        else:       softplus(M, t.index(i), dot)

    M.solve()
```

Logistic regression — example



Logistic regression with increasing regularization. Every point lifted through 28 degree ≤ 6 monomials.

Remark: logistic regression is a (log-)likelihood maximization problem:

$$J(\theta) = \log \prod_i h_\theta(x_i)^{y_i}(1 - h_\theta(x_i))^{1-y_i}.$$

Luxemburg norms

Dirk Lorenz `https://regularize.wordpress.com/2018/05/24/building-norms-from-increasing-and-convex-functions-the-luxemburg-norm/`

$\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ — increasing, convex with $\varphi(0) = 0$. Then the following is a norm on $\mathbb{R}^n$:

$$\|x\|_\varphi = \inf \left\{ \lambda > 0 \ : \ \sum_i \varphi\left(\frac{|x_i|}{\lambda}\right) \leq 1 \right\}.$$

Example: $\varphi(x) = x^p$:

$$\sum_i \left(\frac{|x_i|}{\lambda}\right)^p \leq 1 \iff \lambda \geq \left(\sum_i |x_i|^p\right)^{1/p},$$

so

$$\|x\|_\varphi = \|x\|_p.$$

Observation. The epigraph of the φ–Luxemburg–norm

$$t \geq \|x\|_\varphi$$

is conic representable if the perspective function of φ is.

Proof.

$$\begin{aligned} &w_i \geq |x_i| \\ &s_i \geq t\varphi(w_i/t) \\ &\sum s_i = t \end{aligned}$$

add up to

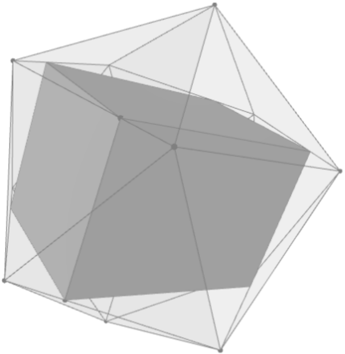
$$1 \geq \sum \varphi(|x_i|/t) \iff t \geq \|x\|_\varphi.$$

Corollary. We can compute with balls in Luxemburg norms for x^p, $x \cdot \log(1+x)$, $\exp(x) - 1$.

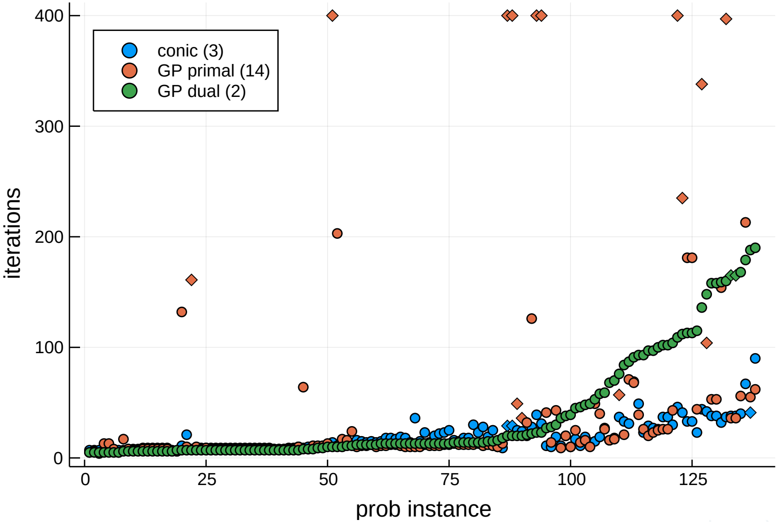
Maximal inscribed cuboid

Find the maximal volume axis-parallel cuboid inscribed in a given convex (conic-representable) set $K \subseteq \mathbb{R}^n$.

$$
\begin{array}{ll}
\text{maximize} & \sum \log d_i \\
\text{s.t.} & x + \varepsilon \circ d \in K, \ \text{for all } \varepsilon \in \{0,1\}^n \\
& x, d \in \mathbb{R}^n.
\end{array}
$$



GP — performance



LogExpCR — performance

Log-exponential convex risk measure, (Vinel, Krokhmal, 2017).

$$
\begin{array}{ll}
\text{minimize} & \eta + (1-\alpha)^{-1} f^{-1}\left(\sum_{j=1}^{m} p_j f(-r_j^T x - \eta)\right) \\
\text{s.t.} & 1^T x \leq 1 \\
 & x^T \sum r_j p_j \geq \bar{r} \\
 & x \in \mathbb{R}^n, \eta \in \mathbb{R}
\end{array}
$$

- generalization of CVaR (Rockafellar, Uryasev, 2002),
- f — vanishing on $\mathbb{R}_-$, $f(0) = 0$, convex on $\mathbb{R}_+$. Here:

$$f(u) = \exp([u]_+) - 1.$$

- n — number of assets.
- m — number of historical scenarios $r_1, \ldots, r_m \in \mathbb{R}^n$ with probabilities $p_1, \ldots, p_m$.

LogExpCR — performance

Easy instances

n	m	8	9
200	100	0.08 (20)	**0.05** (22)
200	200	**0.17** (21)	0.19 (25)
200	500	0.91 (31)	**0.35** (27)
200	1000	4.08 (28)	**0.57** (27)
200	2000	3.32 (39)	**0.99** (28)
500	100	0.13 (20)	**0.11** (23)
500	200	**0.28** (20)	0.36 (27)
500	500	1.61 (34)	**1.41** (31)
500	1000	5.92 (29)	**1.56** (30)
500	2000	25.25 (34)	**2.44** (30)
1000	100	**0.21** (22)	0.21 (29)
1000	200	**0.42** (20)	0.59 (30)
1000	500	3.03 (34)	**2.53** (31)
1000	1000	9.43 (31)	**6.87** (35)
1000	2000	35.26 (32)	**8.66** (32)
1500	100	0.24 (18)	**0.20** (23)
1500	200	**0.62** (20)	0.82 (31)
1500	500	4.11 (35)	**3.99** (33)
1500	1000	16.39 (33)	**10.42** (37)
1500	2000	45.67 (31)	**12.15** (34)

Numerically harder instances

n	m	8	9
200	100	0.12 (23)	**0.06** (29)
200	200	0.42 (67)	**0.29** (37)
200	500	1.12 (43)	**0.77** (59)
200	1000	6.01 (51)	**1.83** (71)
200	2000		**3.44** (87)
500	100		**0.09** (24)
500	200	**0.35** (27)	0.37 (31)
500	500		**2.08** (44)
500	1000	8.12 (46)	**4.45** (80)
500	2000		**5.84** (64)
1000	100	0.31 (38)	**0.13** (22)
1000	200	**0.51** (27)	0.58 (28)
1000	500	3.66 (43)	**3.23** (40)
1000	1000	**12.32** (44)	12.83 (66)
1000	2000		**16.78** (70)
1500	100	0.31 (24)	**0.18** (22)
1500	200	2.08 (83)	**0.70** (28)
1500	500		**6.04** (51)
1500	1000		**11.65** (42)
1500	2000	73.21 (52)	**24.77** (67)

time in sec. (intpnt. iterations)

Closing remarks

Software:

- CVXPY has a $K_{\exp}$–capable MOSEK interface (Riley Murray).
- Also YALMIP.
- MOSEK Version 9 release this year.

Links:

- WWW `www.mosek.com`
- Demos `github.com/MOSEK/Tutorials`
- Blog `themosekblog.blogspot.com/`
- I found a bug! / MOSEK is too slow! `support@mosek.com`
- Twitter `@mosektw`
- Modeling Cookbook `www.mosek.com/documentation`
- Slides: `www.mosek.com/resources/presentations`

Reading:

- V.Chandrasekaran, P.Shah, *Relative entropy optimization and its applications*, Math. Program., Ser. A (2017) 161:1-32

Thank you!

Smallest enclosing ball of a random point set in $\mathbb{R}^2$ in the $(\exp(x) - 1)$–Luxemburg norm.