The Feasibility Pump heuristic for Mixed-Integer Conic Programming

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Mixed-Integer Conic Optimization

We consider problems of the form

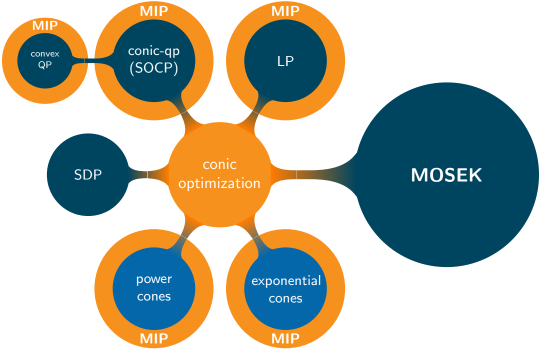
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \in \mathcal{K} \cap \left(\mathbb{Z}^p \times \mathbb{R}^{n-p}\right), \end{array}$$

where $\mathcal{K}$ is a convex cone.

Typically, $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_K$ is a product of lower-dimensional cones - so-called conic building blocks.

What is **MOSEK** ?

MOSEK is a software package for large-scale (Mixed-Integer) Conic Optimization.



Symmetric cones (supported by **MOSEK** 8)

- *the nonnegative orthant*

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_j \geq 0,\ j = 1, \ldots, n\},$$

- *the quadratic cone*

$$\mathcal{Q}^n = \{x \in \mathbb{R}^n \mid x_1 \geq \left(x_2^2 + \cdots + x_n^2\right)^{1/2}\},$$

- *the rotated quadratic cone*

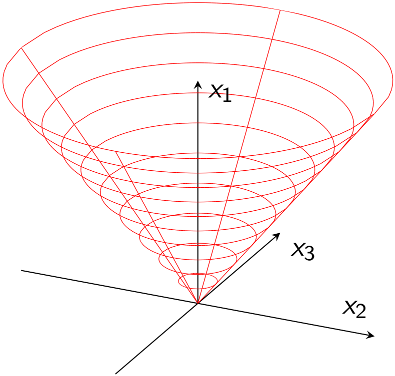
$$\mathcal{Q}^n_r = \{x \in \mathbb{R}^n \mid 2x_1x_2 \geq x_3^2 + \ldots x_n^2,\ x_1, x_2 \geq 0\}.$$

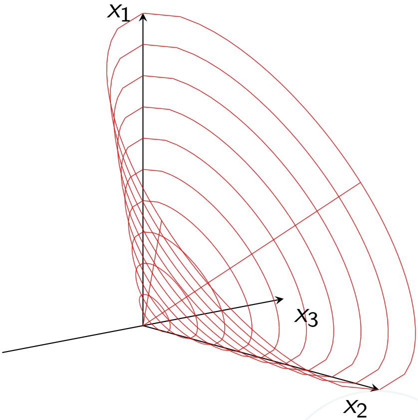
- *the semidefinite matrix cone*

$$\mathcal{S}^n = \{x \in \mathbb{R}^{n(n+1)/2} \mid z^T \mathbf{mat}(x) z \geq 0,\ \forall z\},$$

with $\mathbf{mat}(x) := \left[\begin{array}{cccc} x_1 & x_2/\sqrt{2} & \ldots & x_n/\sqrt{2} \\ x_2/\sqrt{2} & x_{n+1} & \ldots & x_{2n-1}/\sqrt{2} \\ \vdots & \vdots & & \vdots \\ x_n/\sqrt{2} & x_{2n-1}/\sqrt{2} & \ldots & x_{n(n+1)/2} \end{array}\right].$

Quadratic cones in dimension 3





Non-symmetric cones (in next **MOSEK** release)

- *the three-dimensional power cone*

$$\mathcal{P}^{\alpha} = \{x \in \mathbb{R}^3 \mid x_1^{\alpha} x_2^{(1-\alpha)} \geq |x_3|,\ x_1, x_2 \geq 0\},$$

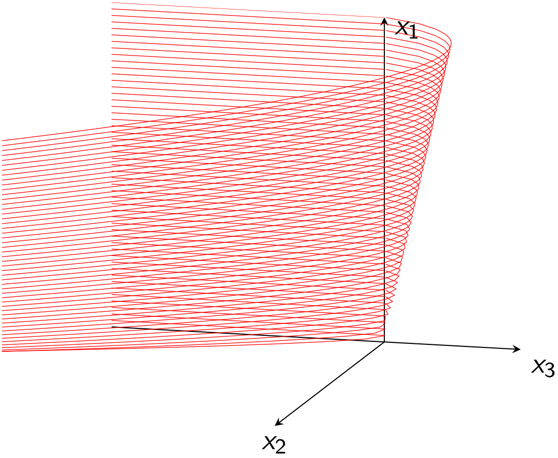
for $0 < \alpha < 1$.

- *the three-dimensional exponential cone*

$$\mathcal{K}_{exp} = \mathrm{cl}\{x \in \mathbb{R}^3 \mid x_1 \geq x_2 \exp(x_3/x_2),\ x_2 > 0\}.$$

Interior-point methods for non-symmetric cones are less studied, and less mature.

The exponential cone



The beauty of Conic Optimization

In continuous optimization, conic (re-)formulations have been highly advocated for quite some time, e.g., by Nemirovski [13].

- Separation of data and structure:
  - Data: c, A and b.
  - Structure: $\mathcal{K}$.
- Structural convexity.
- Duality (almost...).
- No issues with smoothness and differentiability.

We call modeling with the aforementioned 5 cones **extremely disciplined convex programming**: "Almost all convex constraints which arise in practice are representable by using these cones."

Cones in Mixed-Integer Optimization

Lubin et al. [11] show that all convex instances (333) in MINLPLIB2 are conic representable using only 4 types of cones.

The exploitation of conic structures in the mixed-integer case is slightly newer, but nonetheless an active research area:

- MISOCP:
  - Extended Formulations: Vielma et al. [14].
  - Cutting planes: Andersen and Jensen [1], Kılınç-Karzan and Yıldız [9], Belotti et al. [2], ...
  - Primal heuristics: Çay, Pólik and Terlaky [5].
- Duality: Morán, Dey and Vielma [12].
- Outer approximation: Lubin [10].
- ...

Mixed-integer optimization in **MOSEK**

- **MOSEK** allows mixed-integer variables in combination with the linear, the conic-quadratic, the exponential and the power cones.
- Applies a branch-and-cut/branch-and-bound framework.
- Preliminary work in case of the non-symmetric cones.
- Tested on mixed-integer exp-cone instances from CBLIB by Miles Lubin.

Mixed-integer exponential-cone instances I
Successfully solved instances

	Time	Obj. value	# nodes
syn40m04h	6.58	-901.75	476
syn40m03h	2.31	-395.15	276
syn40m02h	0.43	-388.77	14
syn40h	0.19	-67.713	16
syn30m04h	3.27	-865.72	450
syn30m03h	1.11	-654.16	165
syn30m02m	1091.4	-399.68	348085
syn30m02h	0.44	-399.68	58
syn30m	9.98	-138.16	7849
syn30h	0.13	-138.16	11
syn20m04m	1833.48	-3532.7	534769
syn20m04h	0.55	-3532.7	27
syn20m03m	300.47	-2647	118089
syn20m03h	0.37	-2647	25
syn20m02m	28.21	-1752.1	14321
syn20m02h	0.19	-1752.1	11
syn20m	0.63	-924.26	645
syn20h	0.09	-924.26	11
syn15m04m	16.59	-4937.5	5567
syn15m04h	0.33	-4937.5	7
syn15m03m	4.77	-3850.2	1907
syn15m03h	0.19	-3850.2	5
syn15m02m	1.24	-2832.7	751
syn15m02h	0.11	-2832.7	5
syn15m	0.12	-853.28	85
syn15h	0.04	-853.28	3
syn10m04m	2.99	-4557.1	1983
syn10m04h	0.16	-4557.1	5

Mixed-integer exponential-cone instances II

Successfully solved instances

syn10m03m	1.13	-3354.7	923
syn10m03h	0.11	-3354.7	5
syn10m02m	0.36	-2310.3	409
syn10m02h	0.08	-2310.3	5
syn10m	0.05	-1267.4	31
syn10h	0	-1267.4	0
syn05m04m	0.17	-5510.4	45
syn05m04h	0.06	-5510.4	3
syn05m03m	0.09	-4027.4	33
syn05m03h	0.04	-4027.4	3
syn05m02m	0.06	-3032.7	23
syn05m02h	0.03	-3032.7	3
syn05m	0.02	-837.73	11
syn05h	0.02	-837.73	5
rsyn0840m04h	39.28	-2564.5	2197
rsyn0840m03h	15.34	-2742.6	1577
rsyn0840m02h	1.56	-734.98	149
rsyn0840h	0.27	-325.55	19
rsyn0830m04h	29.9	-2529.1	2115
rsyn0830m03h	8.3	-1543.1	935
rsyn0830m02h	2.38	-730.51	299
rsyn0830m	227.14	-510.07	99495
rsyn0830h	0.44	-510.07	117
rsyn0820m04h	10.59	-2450.8	635
rsyn0820m03h	18.16	-2028.8	2079
rsyn0820m02h	3.35	-1092.1	510
rsyn0820m	110.08	-1150.3	58607
rsyn0820h	0.46	-1150.3	145
rsyn0815m04h	5.79	-3410.9	587
rsyn0815m03h	7.37	-2827.9	866

Mixed-integer exponential-cone instances III

Successfully solved instances

rsyn0815m02m	2345.68	-1774.4	567030
rsyn0815m02h	2.08	-1774.4	365
rsyn0815m	10.47	-1269.9	7059
rsyn0815h	0.36	-1269.9	238
rsyn0810m04h	6.95	-6581.9	677
rsyn0810m03h	4.95	-2722.4	740
rsyn0810m02m	1353.22	-1741.4	425403
rsyn0810m02h	1.15	-1741.4	159
rsyn0810m	8.31	-1721.4	9041
rsyn0810h	0.21	-1721.4	134
rsyn0805m04m	578.5	-7174.2	66975
rsyn0805m04h	1.92	-7174.2	101
rsyn0805m03m	186.01	-3068.9	37908
rsyn0805m03h	1.61	-3068.9	177
rsyn0805m02m	86.81	-2238.4	34126
rsyn0805m02h	0.87	-2238.4	201
rsyn0805m	3.16	-1296.1	4639
rsyn0805h	0.19	-1296.1	120

Timed-out instances

	Time	Obj. value	# nodes
gams01	3600.0	22265	70232
rsyn0810m03m	3600.0	-2722.4	493926
rsyn0810m04m	3600.0	-6580.9	307231
rsyn0815m03m	3600.1	-2827.9	420782
rsyn0815m04m	3600.2	-3359.8	309729
rsyn0820m02m	3600.2	-1077.6	683356
rsyn0820m03m	3600.2	-1980.4	380611
rsyn0820m04m	3600.1	-2401.1	262880
rsyn0830m02m	3600.4	-705.46	568113
rsyn0830m03m	3600.2	-1456.3	368794
rsyn0830m04m	3600.1	-2395.7	206456
rsyn0840m	3600.3	-325.55	1157426
rsyn0840m02m	3600.5	-634.17	422224
rsyn0840m03m	3600.1	-2656.5	252651
rsyn0840m04m	3600.0	-2426.3	142895
syn30m03m	3600.2	-654.15	831798
syn30m04m	3600.2	-848.07	643266
syn40m02m	3600.2	-366.77	748603
syn40m03m	3600.3	-355.64	607359
syn40m04m	3600.2	-859.71	371521

WIP: Exploiting conic structures in FP

For convex MINLP, two variants of the Feasibility Pump heuristic have been proposed:

- A straightforward extension of the original scheme in [6] by solving convex NLPs in the projection step [4].
- A similar extension with an additional elaboration of the rounding step [3].

In this talk, we focus on the first variant:

algorithm: `fp-convex`

$C := \{x : Ax = b, x \in \mathcal{K}\}$;
$x^* = \arg\min\{c^T x : x \in C\}$;
while *not termination criterion* **do**
  if x^* *is integer* **then** return x^*;
  $\tilde{x} = \texttt{Round}(x^*)$;
  if *cycle detected* **then** $\texttt{Perturb}(\tilde{x})$;
  $x^* = \texttt{Project}_C(\tilde{x})$;
end

WIP: Exploiting conic structures in FP (cont.)

Two observations:

- When extending FP from linear to non-linear problems, we cannot use the simplex algorithm any longer!
- FP is a successive-projection method, and it is usually quite easy to project onto cones.

Idea: shift the satisfaction of conic constraints from the projection step to the rounding step!

algorithm: `fp-conic`

$P := \{x : Ax = b, x_L \geq 0\}$ // $L = \{i : \mathrm{proj}_{x_i}(\mathcal{K}) = \mathbb{R}_+\}$;
$x^* = \arg\min\{c^T x : x \in P\}$;
while *not termination criterion* **do**
  if $x^* \in \mathcal{K} \cap \left(\mathbb{Z}^p \times \mathbb{R}^{n-p}\right)$ **then** return x^*;
  $\tilde{x} = \texttt{ConicRound}_{\mathcal{K}}(x^*)$;
  if *cycle detected* **then** $\texttt{Perturb}(\tilde{x})$;
  $x^* = \texttt{Project}_P(\tilde{x})$;
end

WIP: Exploiting conic structures in FP (cont.)

Instead of generating a sequence $\{(x^*, \tilde{x})\}_k$ in

$$C \times \left(\mathbb{Z}^p \times \mathbb{R}^{(n-p)}\right),$$

we try to generate one in

$$P \times \left(\mathcal{K} \cap \left(\mathbb{Z}^p \times \mathbb{R}^{(n-p)}\right)\right).$$

Then we can solve the projections onto P as LPs, in particular we can use warm-stars.

In turn, the procedure $\texttt{ConicRound}_{\mathcal{K}}(\cdot)$ has to transform the point x^* into an integral point that additionally satisfies the conic constraints.

This can be achieved by exploiting cone projections.

Cone projections

When dealing with cones, it is often desirable to solve the projection problem

$$p' = \arg\min\{\|x - p\|_2 : x \in \mathcal{K}\}$$

for some cone $\mathcal{K} \subseteq \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$.

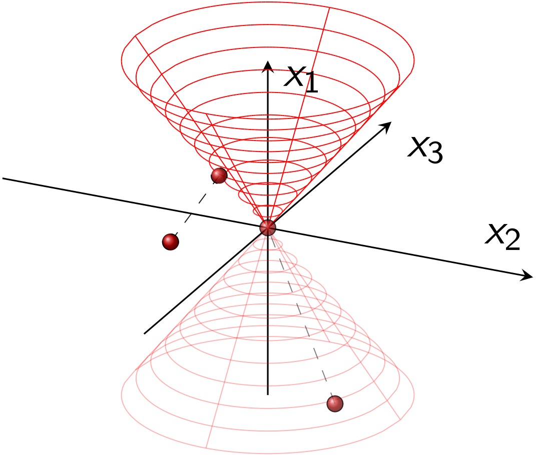
In some cases, this is possible analytically:

- If $p \in \mathbb{R}$ and $\mathcal{K} = \mathbb{R}_+$, then $p' = \max(0, p)$.
- If $p = (t, s) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $\mathcal{K} = \mathcal{Q}^n$, then

$$p' = \begin{cases} (t, s), & t \geq \|s\|_2 \\ \frac{1}{2}\left(\frac{t}{\|s\|_2} + 1\right) \cdot (\|s\|_2, s), & -\|s\|_2 < t < \|s\|_2 \\ 0, & t \leq -\|s\|_2. \end{cases}$$

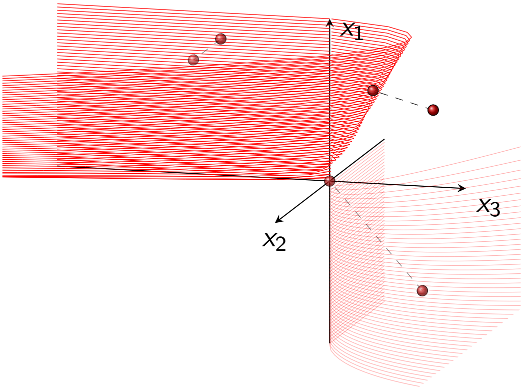
- A symmetric matrix can be projected onto the semidefinite cone analytically via its spectral decomposition.

Projecting onto the quadratic cone



Cone projections (cont.)

For the exponential and power cones, the projection problem is at most a univariate root-finding problem [8, 7].



Combining rounding with cone projections

Note that every variable can belong to at most one cone!

In order to implement $\texttt{ConicRound}_{\mathcal{K}}(\cdot)$, two ways are thinkable:

- Assume w.l.o.g. that $\{1, \ldots, p\} \subseteq L = \{i : \mathrm{proj}_{x_i}(\mathcal{K}) = \mathbb{R}_+\}$. Integer variables are rounded, continuous variables are projected onto their cones.
- Apply S-preserving roundings.

Definition

Let $S \subseteq \mathbb{R}^n$. We call a function $r : \mathbb{R}^n \to \mathbb{Z}^p \times \mathbb{R}^{(n-p)}$ an S-preserving rounding, iff $r(S) \subseteq S$.

Combining rounding with cone projections (cont.)

With such a rounding, each variable can first be projected onto its cone, and then rounded, if necessary.

Example

If $S = \mathcal{Q}^n$, then $r(x) = (\lceil x_1 \rceil, \lfloor x_2 \rfloor, \ldots, \lfloor x_n \rfloor)^T$ is S-preserving.

However, the practical relevance of such roundings is unclear: most variables belonging to non-linear cones are continuous.

Note that for the perturbation step, amendments similar to those for the rounding step are desirable.

Computation: MISOCP

		fp-convex			fp-conic		
		time	#it	success	time	#it	success
cb_robust	(n=35)	3.43	1.0	94.8%	1.0	2.30	73.8%
cb_shortfall	(n=56)	1.74	1.0	96.2%	1.0	4.54	56.7%
cb_classical	(n=14)	1.42	1.0	98.2%	1.0	2.91	60.5%

- Significant speed-ups can be observed.
- In some cases, numerical issues arising in `fp-convex` are circumvented.
- In this basic version however, the advantages are not consistent and seemingly limited to instances with few cones...

Open issues

- `fp-conic` exhibits non-negligible convergence problems. Even when the algorithm converges, it would be desirable to reduce the number of iterations.
- Can $\texttt{ConicRound}_{\mathcal{K}}(\cdot)$ be integrated with Outer-approximation cuts?
- How does `fp-conic` behave when dealing with power or exponential cones?
- How does `fp-conic` behave on hard instances?

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