Power cones in second-order cone form and dual recovery

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Common restrictions

- $\sum_1^k \alpha_j = \mathrm{e}^\mathsf{T}\alpha = 1$.
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- $\alpha \in \mathbb{R}^k_{++}$.
  Full generality by $\mathcal{P}^n_{(0,\alpha)} = \mathbb{R}_+ \times \mathcal{P}^n_\alpha$. When are zeros useful?
  - Powers, $s \geq |x|^p$, for any $p \geq 1$:
    $$(1, s, x) \in \mathcal{P}^3_{(p-1),1} \iff 1^{p-1}s^1 \geq |x|^p$$
  - p-norms, $t \geq \|x\|_p$, for any $p \geq 1$:
    $$t \geq \mathrm{e}^\mathsf{T}s, \quad \text{and} \quad (t, s_j, x_j) \in \mathcal{P}^3_{(p-1),1} \quad \forall j$$

What is a power cone?

The **dual power cone** was be obtained on $\alpha \subseteq \mathbb{R}^k_{++}$ by (Chares 2009, Theorem 4.3.1) as:

$$(\mathcal{P}^n_\alpha)^* = M\mathcal{P}^n_\alpha, \quad \text{for } M = \begin{pmatrix} (\mathsf{e}^\mathsf{T}\alpha)^{-1}\,\mathsf{diag}(\alpha) & 0 \\ 0 & \mathrm{I_{n-k}} \end{pmatrix} \succ 0,$$

expanding to:

$$(\mathcal{P}^n_\alpha)^* = \{(x, z) \in \mathbb{R}^k_+ \times \mathbb{R}^{n-k} \mid \alpha^{-\alpha} x^\alpha \geq (\mathsf{e}^\mathsf{T}\alpha)^{-\mathsf{e}^\mathsf{T}\alpha} \|z\|_2^{\mathsf{e}^\mathsf{T}\alpha}\},$$

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Note self-duality of $M^{1/2}\mathcal{P}^n_\alpha$ in general (the self-dualized variant).

Power cones in MOSEK?

Power cones in MOSEK? Absolutely***!

1. Convert α to rationals. Best rational approximations to π:
$\frac{3}{1}, \frac{13}{4}, \frac{16}{5}, \frac{19}{6}, \boldsymbol{\frac{22}{7}}, \frac{179}{57}, \frac{201}{64}, \frac{223}{71}, \frac{245}{78}, \frac{267}{85}, \frac{289}{92}, \frac{311}{99}, \frac{333}{106}, \boldsymbol{\frac{355}{113}}, \frac{52163}{16604}, \cdots$

2. Use $\mathcal{P}_{\alpha}^{n} = \mathcal{P}_{\lambda\alpha}^{n}$ with $\lambda = \frac{\mathrm{lcm(denominators)}}{\mathrm{gcd(numerators)}}$ to make α integer.

3. Construct tower of variables (Ben-tal and Nemirovski 2001); here $x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 \geq \omega_1^8$.

$$x_1 x_2 \geq y_1^2$$

$$y_1^2 y_2^2 \geq z_1^4$$

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Non-unique, e.g. permute x

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$\boxed{\text{Distinct, e.g., consider } x_1 = x_2}$

Complication summary

- Implementation: cumbersome and error-prone
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Same three complications decomposing $\mathcal{P}^{k+1}_{(\alpha_1,\ldots,\alpha_k)}$ into $k-1$ power cones of the form $\mathcal{P}^{3}_{(\alpha_1,\alpha_2)}$. See Chares (2009). Reason?

- ☹ Barrier parameter increases.
- ☺ Linear outer approximation is stronger.
- ☺ Hessian matrix is approximated with less effort in quasi-newton methods, e.g., using BFGS updates.

Start with any power cone defined by $\alpha \in \mathbb{R}_+^k$:

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Rules:

1. α is invariant to permutation, zeros and positive scaling.

Let's play Tower Tycoon

Rules of the game

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Split rule

$$\begin{aligned} x^\alpha \geq \|z\|_2^{\mathrm{e}^\mathsf{T}\alpha} \quad &\Leftrightarrow \quad x^{\alpha-\beta} x^\beta \geq \|z\|_2^{\mathrm{e}^\mathsf{T}\alpha}, \\ &\Leftrightarrow \quad x^{\alpha-\beta} u^{\mathrm{e}^\mathsf{T}\beta} \geq \|z\|_2^{\mathrm{e}^\mathsf{T}\alpha}, \quad x^\beta \geq u^{\mathrm{e}^\mathsf{T}\beta} \end{aligned}$$

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simple base

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Let's play Tower Tycoon

Goal: second-order cone representation

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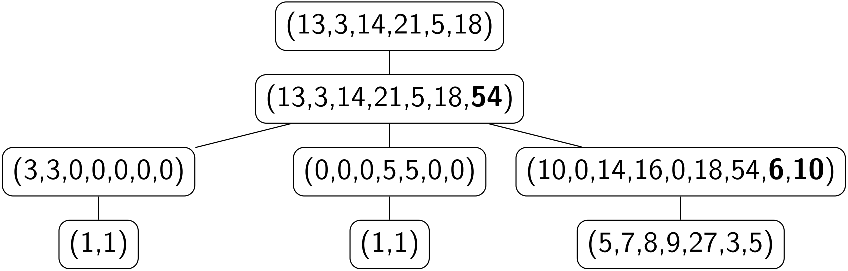
Objective: Transform α to a set of second-order representable power cone parameters, minimizing the number of cones.

- Split rule costs 1 cone.
- Expand rule costs 0 cones on simple base, 1 otherwise.

Let's play Tower Tycoon

Strategy: Powers of 2

(Morenko et al. 2013) worked on, and proved their strategy optimal for, cone $\mathcal{P}^3_{(\alpha_1,\alpha_2)}$ *with simple base. Generalized here.*



1. Initialize: $2^6 < \mathrm{e}^{\mathsf{T}}(13, 3, 14, 21, 5, 18) < 2^7$ with **54** to upper.
2. $\mathrm{e}^{\mathsf{T}}(13, 3, 14, 21, 5, 18, 54) = 2^7$.

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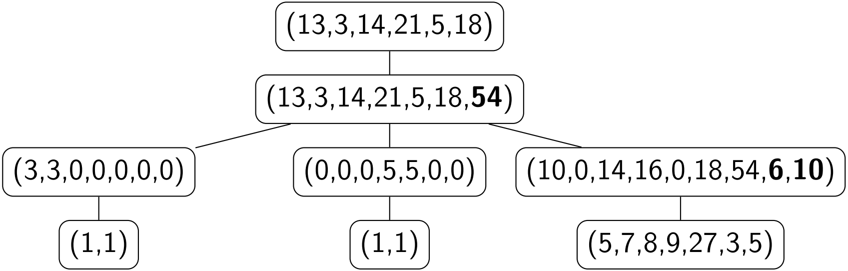

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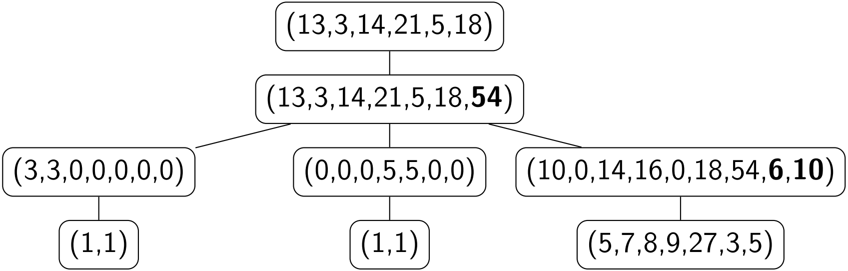

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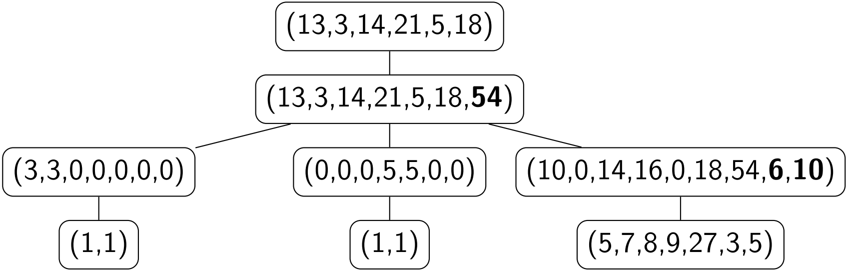

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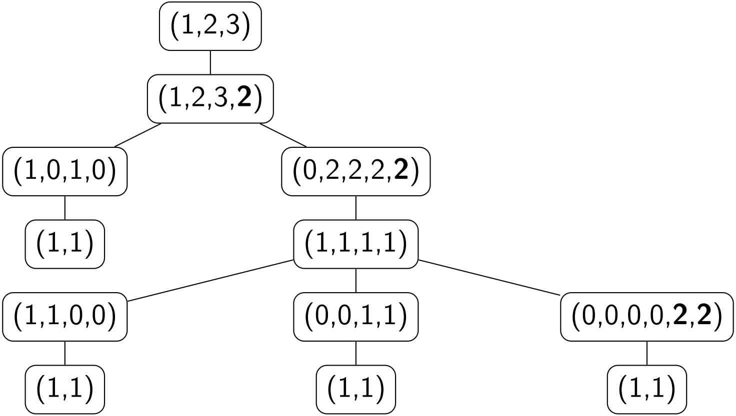
3. Apply split rule to odd power pairs (in this case 2 pairs).
4. $\mathrm{e}^\mathsf{T}(5,7,8,9,27,3,5) = 2^6$.
5. Apply split rule to odd power pairs (in this case 3 pairs).
6. $\mathrm{e}^\mathsf{T}(1,4,9,1,5,9,3) = 2^5$...

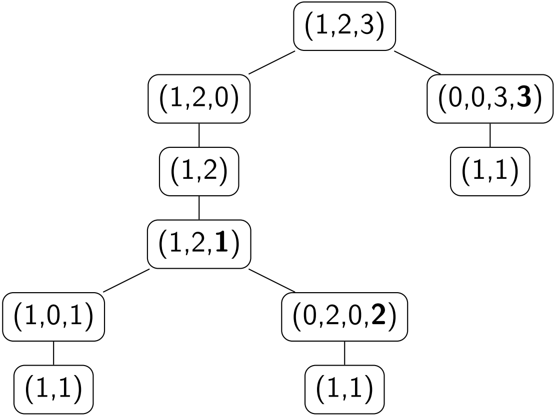
Let's play Tower Tycoon
Still room for improvement



$|S| = 4$ if initial cone has a simple base, and $|S| = 5$ otherwise.

Let's play Tower Tycoon

Still room for improvement (subset sum split)



$|S| = 3.$

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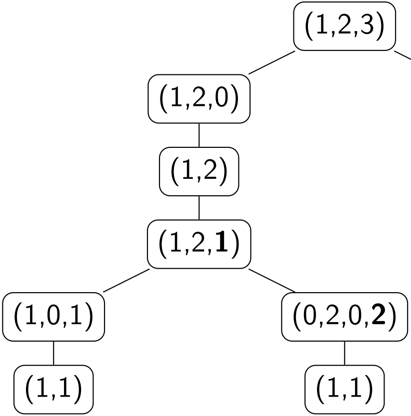

$|S| = 3$. In fact, subset sum splits handle $(1, 2, 3, 6, 12, 24, 48, \ldots)$ in k second-order cones, while the powers of 2 strategy (empirically) uses $2(k-1)$ second-order cones.

Dual information recovery

Split rule

$$
\begin{aligned}
x^{\alpha} \geq \|z\|_2^{e^T\alpha} \quad &\Leftrightarrow \quad x^{\alpha-\beta} x^{\beta} \geq \|z\|_2^{e^T\alpha}, \\
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Expansion rule

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&\Leftrightarrow \quad x^{\alpha} u^{\beta} \geq u^{e^T\alpha+\beta}, \quad u \geq \|z\|_2,
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$$

Dual information recovery

Split rule

	BEFORE	**AFTER**
PRIMAL	$\binom{x}{z} \in \mathcal{P}^n_\alpha \quad \begin{bmatrix} s \\ t \end{bmatrix}$	$\begin{pmatrix} x \\ x \\ z \end{pmatrix} \in \mathcal{P}^n_{(\alpha-\beta,\beta)} \quad \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau \end{bmatrix}$
DUAL	$x: \quad +s$ $z: \quad +t$ where $(s,t) \in \left(\mathcal{P}^n_\alpha\right)^*$	$x: \quad +\sigma_1 + \sigma_2$ $z: \quad +\tau$ where $(\sigma_1, \sigma_2, \tau) \in \left(\mathcal{P}^n_{(\alpha-\beta,\beta)}\right)^*$

Recover as $(s,t) \leftarrow (\sigma_1 + \sigma_2, \tau)$.

Dual information recovery

Expansion rule

	BEFORE	AFTER
PRIMAL	$\binom{x}{u} \in \mathcal{P}_{\alpha}^{n_x+1}$ $\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]$ $(u \geq 0)$	$\begin{pmatrix} x \\ u \\ u \end{pmatrix} \in \mathcal{P}_{(\alpha,\beta)}^{n_x+2}$ $\begin{bmatrix} \sigma \\ \tau_1 \\ \tau_2 \end{bmatrix}$ $(u \geq 0)$
DUAL	$x: \quad +s$ $u: \quad +t \; (\leq 0)$ where $(s,t) \in \left(\mathcal{P}_{\alpha}^{n_x+1}\right)^*$	$x: \quad +\sigma$ $u: \quad +\tau_1 + \tau_2 \; (\leq 0)$ where $(\sigma, \tau_1, \tau_2) \in \left(\mathcal{P}_{(\alpha,\beta)}^{n_x+2}\right)^*$

Recover as $(s,t) \leftarrow (\sigma, \tau_1 + \tau_2)$.

Dual information recovery

Prerequisites of an elegant proof

Dual split rule

$$
\begin{aligned}
(x, z) \in (\mathcal{P}_{\alpha}^{n})^{*} \quad &\Leftrightarrow \quad (x-u, u, z) \in \left(\mathcal{P}_{(\alpha-\beta,\beta)}^{n}\right)^{*}, \\
&\Leftrightarrow \quad (x-u, v, z) \in \left(\mathcal{P}_{(\alpha-\beta,\mathsf{e}^{\mathsf{T}}\beta)}^{n}\right)^{*}, (u, v) \in \left(\mathcal{P}_{\mathsf{e}^{\mathsf{T}}\beta}^{n}\right)^{*}.
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Dual expansion rule

$$
(x, z) \in (\mathcal{P}_{\alpha}^{n_x+1})^{*},\ z \geq 0 \quad \Leftrightarrow \quad (x, u, z+u) \in \left(\mathcal{P}_{(\alpha,\beta)}^{n_x+2}\right)^{*}.
$$

Dual information recovery

Proving the prerequisites

The AM-GM inequality does it all:

$$(\mathsf{e}^{\mathsf{T}}\alpha)^{-1}(\alpha^T x) \geq \sqrt[\mathsf{e}^{\mathsf{T}}\alpha]{x^\alpha},$$

for $x, \alpha \in \mathbb{R}_+^k$ where $\mathsf{e}^{\mathsf{T}}\alpha > 0$.

Bonus info

It gives rise to a family of outer approximations, the simplest of which is a quadratic cone:

$$\mathcal{P}_\alpha^n \subseteq \{(x, z) \in \mathbb{R}_+^k \times \mathbb{R}^{n-k} \mid (\mathsf{e}^{\mathsf{T}}\alpha)^{-1}(\alpha^T x) \geq \|z\|_2\},$$

Numerical results

Shooting sparrows with a cannon

The 8'th root of 42 is 1.5955343603, but also the infimum of

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & y = 42, \\ & (y, 1, x) \in \mathcal{P}^3_{(1,7)}. \end{array}$$

ITE	PFEAS	DFEAS	GFEAS	PRSTATUS	POBJ	DOBJ	MU	TIME
0	5.5e+00	1.0e+00	1.0e+00	0.00e+00	0.000000000e+00	0.000000000e+00	1.0e+00	0.01
1	1.1e+00	2.1e-01	1.5e-01	-6.56e-01	2.184849059e-01	-1.207084580e+00	2.1e-01	0.01
2	2.2e-01	3.9e-02	5.4e-02	3.82e-01	5.765513222e-01	1.852211210e-01	3.9e-02	0.01
3	4.2e-02	7.8e-03	2.0e-02	7.43e-01	1.340272353e+00	1.221223568e+00	7.8e-03	0.01
4	7.2e-03	1.3e-03	7.7e-03	8.65e-01	1.539177880e+00	1.515646623e+00	1.3e-03	0.01
5	3.1e-04	5.6e-05	1.6e-03	9.55e-01	1.593269995e+00	1.592202275e+00	5.6e-05	0.01
6	7.0e-06	1.3e-06	2.3e-04	9.98e-01	1.595487015e+00	1.595462738e+00	1.3e-06	0.01
7	2.6e-07	4.8e-08	4.5e-05	1.00e+00	1.595532790e+00	1.595531871e+00	4.8e-08	0.01
8	1.6e-08	2.9e-09	1.1e-05	1.00e+00	1.595534274e+00	1.595534219e+00	2.9e-09	0.01

```
Optimizer terminated. Time: 0.03

Interior-point solution summary
  Problem status  : PRIMAL_AND_DUAL_FEASIBLE
  Solution status : OPTIMAL
  Primal.  obj: 1.5955342736e+00    nrm: 4e+01    Viol.  con: 9e-09    var: 0e+00    cones: 0e+00
  Dual.    obj: 1.5955342195e+00    nrm: 1e+00    Viol.  con: 0e+00    var: 1e-08    cones: 3e-09
```

Two quadratic cones after presolve. Complementarity is $x^T s = 3.388688\mathrm{e}{-08}$ after dual information recovery.