Power cones in second-order cone form and dual recovery

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- Geometric mean:
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  $$\mathcal{P}_{1,1,\ldots,1}^n = \{(x,z) \mid x_1^1 x_2^1 \cdots x_k^1 \geq \|z\|_2^k\}$$
- Weighted geometric mean:
  $$\mathcal{P}_{\alpha_1,\alpha_2,\ldots,\alpha_k}^n = \{(x,z) \mid x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} \geq \|z\|_2^{\alpha_1+\alpha_2+\ldots+\alpha_k}\}$$

What is a power cone?

Defined by parameter vector, $\alpha \in \mathbb{R}^k_+$, spanning:

- Quadratic cone:
$$\mathcal{P}^n_1 \quad = \{(x, z) \mid x_1^1 \geq \|z\|_2\}$$
- Rotated quadratic cone in the non-self-dualized form:
$$\mathcal{P}^n_{1,1} \quad = \{(x, z) \mid x_1^1 x_2^1 \geq \|z\|_2^2\}$$
- Geometric mean:
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- Weighted geometric mean:
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The **power cone** can be given for any $\alpha \in \mathbb{R}^k_+$ as
$$\mathcal{P}^n_\alpha = \{(x, z) \in \mathbb{R}^k_+ \times \mathbb{R}^{n-k} \mid x^\alpha \geq \|z\|_2^{e^T\alpha}\},$$
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Common restrictions

- $\sum_1^k \alpha_j = \mathrm{e}^\mathsf{T}\alpha = 1$.
  Full generality by scale invariance $\mathcal{P}^n_\alpha = \mathcal{P}^n_{\lambda\alpha}$ for $\lambda > 0$, but only useful in barrier function to my knowledge.

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  Full generality by scale invariance $\mathcal{P}^n_\alpha = \mathcal{P}^n_{\lambda\alpha}$ for $\lambda > 0$, but only useful in barrier function to my knowledge.
- $\alpha \in \mathbb{R}^k_{++}$.
  Full generality by $\mathcal{P}^n_{(0,\alpha)} = \mathbb{R}_+ \times \mathcal{P}^n_\alpha$. When are zeros useful?
  - Powers, $s \geq |x|^p$, for any $p \geq 1$:
    $$(1, s, x) \in \mathcal{P}^3_{(p-1),1} \iff 1^{p-1}s^1 \geq |x|^p$$
  - p-norms, $t \geq \|x\|_p$, for any $p \geq 1$:
    $$t \geq e^T s, \quad \text{and} \quad (t, s_j, x_j) \in \mathcal{P}^3_{(p-1),1} \;\; \forall j$$

What is a power cone?

The **dual power cone** was be obtained on $\alpha \subseteq \mathbb{R}^k_{++}$ by (Chares 2009, Theorem 4.3.1) as:

$$(\mathcal{P}^n_\alpha)^* = M\mathcal{P}^n_\alpha, \quad \text{for } M = \begin{pmatrix} (\mathsf{e}^\mathsf{T}\alpha)^{-1}\operatorname{diag}(\alpha) & 0 \\ 0 & \mathrm{I_{n-k}} \end{pmatrix} \succ 0,$$

expanding to:

$$(\mathcal{P}^n_\alpha)^* = \{(x, z) \in \mathbb{R}^k_+ \times \mathbb{R}^{n-k} \mid \alpha^{-\alpha}x^\alpha \geq (\mathsf{e}^\mathsf{T}\alpha)^{-\mathsf{e}^\mathsf{T}\alpha}\|z\|_2^{\mathsf{e}^\mathsf{T}\alpha}\},$$

which is easily shown valid on all of $\alpha \subseteq \mathbb{R}^k_+$.

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Note self-duality of $M^{1/2}\mathcal{P}^n_\alpha$ in general (the self-dualized variant).

Power cones in MOSEK?

Power cones in MOSEK? Absolutely***!

1. Convert α to rationals. Best rational approximations to π:
$\frac{3}{1}, \frac{13}{4}, \frac{16}{5}, \frac{19}{6}, \mathbf{\frac{22}{7}}, \frac{179}{57}, \frac{201}{64}, \frac{223}{71}, \frac{245}{78}, \frac{267}{85}, \frac{289}{92}, \frac{311}{99}, \frac{333}{106}, \mathbf{\frac{355}{113}}, \frac{52163}{16604}, \cdots$

2. Use $\mathcal{P}^n_\alpha = \mathcal{P}^n_{\lambda\alpha}$ with $\lambda = \frac{\mathrm{lcm}(\text{denominators})}{\mathrm{gcd}(\text{numerators})}$ to make α integer.

3. Construct tower of variables (Ben-tal and Nemirovski 2001); here $x_1x_2x_3x_4x_5x_6x_7x_8 \geq \omega_1^8$.

$$z_1^4 z_2^4 \geq w_1^8$$

$$y_1^2 y_2^2 \geq z_1^4 \qquad y_3^2 y_4^2 \geq z_2^4$$

$$x_1x_2 \geq y_1^2 \qquad x_3x_4 \geq y_2^2 \qquad x_5x_6 \geq y_3^2 \qquad x_7x_8 \geq y_4^2$$

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$\boxed{\text{Non-unique, e.g. permute } x}$

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$\boxed{\text{Distinct, e.g., consider } x_1 = x_2}$

Complication summary

- Implementation: cumbersome and error-prone
- Tower constructions: suboptimal
- Dual information: where?

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Same three complications decomposing $\mathcal{P}^{k+1}_{(\alpha_1,\ldots,\alpha_k)}$ into $k-1$ power cones of the form $\mathcal{P}^{3}_{(\alpha_1,\alpha_2)}$. See Chares (2009). Reason?

- ☹ Barrier parameter increases.
- ☺ Linear outer approximation is stronger.
- ☺ Hessian matrix is approximated with less effort in quasi-newton methods, e.g., using BFGS updates.

Start with any power cone defined by $\alpha \in \mathbb{R}_+^k$:

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Rules:

1. α is invariant to permutation, zeros and positive scaling.

Let's play Tower Tycoon
Rules of the game

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2. Split $\alpha \longrightarrow \Big\{(\alpha - \beta, e^\mathsf{T}\beta),\ (\beta)\Big\}$ for any $\beta \leq \alpha$.

Split rule

$$\begin{aligned} x^\alpha \geq \|z\|_2^{e^\mathsf{T}\alpha} \quad &\Leftrightarrow \quad x^{\alpha-\beta} x^\beta \geq \|z\|_2^{e^\mathsf{T}\alpha}, \\ &\Leftrightarrow \quad x^{\alpha-\beta} u^{e^\mathsf{T}\beta} \geq \|z\|_2^{e^\mathsf{T}\alpha}, \quad x^\beta \geq u^{e^\mathsf{T}\beta} \end{aligned}$$

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Expansion rule

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Expansion rule

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simple base

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4. Expand $\alpha \longrightarrow \{(\alpha, \beta)\}$ for any $\beta \in \mathbb{R}_+$ (on simple base).

Expansion rule

$$\begin{aligned}
x^\alpha \geq \|z\|_2^{e^\mathsf{T}\alpha} \quad &\Leftrightarrow \quad x^\alpha \geq u^{e^\mathsf{T}\alpha} \geq \|z\|_2^{e^\mathsf{T}\alpha}, \\
&\Leftrightarrow \quad x^\alpha \geq u^{e^\mathsf{T}\alpha}, \quad u \geq \|z\|_2, \\
&\Leftrightarrow \quad x^\alpha u^\beta \geq u^{e^\mathsf{T}\alpha + \beta}, \quad u \geq \|z\|_2,
\end{aligned}$$

Let's play Tower Tycoon

Goal: second-order cone representation

Start with any power cone defined by $\boxed{\alpha \in \mathbb{Z}^k_+}$:

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4. Expand $\alpha \longrightarrow \{(\alpha, \beta)\}$ for any $\beta \in \mathbb{R}_+$ (on simple base).

Objective: Transform α to a set of second-order representable power cone parameters, minimizing the number of cones.

- Split rule costs 1 cone.
- Expand rule costs 0 cones on simple base, 1 otherwise.

Let's play Tower Tycoon

Strategy: Powers of 2

(Morenko et al. 2013) worked on, and proved their strategy optimal for, cone $\mathcal{P}^3_{(\alpha_1,\alpha_2)}$ with simple base. Generalized here.

(13,3,14,21,5,18)

(13,3,14,21,5,18,**54**)

(3,3,0,0,0,0,0) — (1,1)

(0,0,0,5,5,0,0) — (1,1)

(10,0,14,16,0,18,54,**6**,**10**) — (5,7,8,9,27,3,5)

1. Initialize: $2^6 < e^T(13,3,14,21,5,18) < 2^7$ with **54** to upper.
2. $e^T(13,3,14,21,5,18,54) = 2^7$.

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3. Apply split rule to odd power pairs (in this case 2 pairs).

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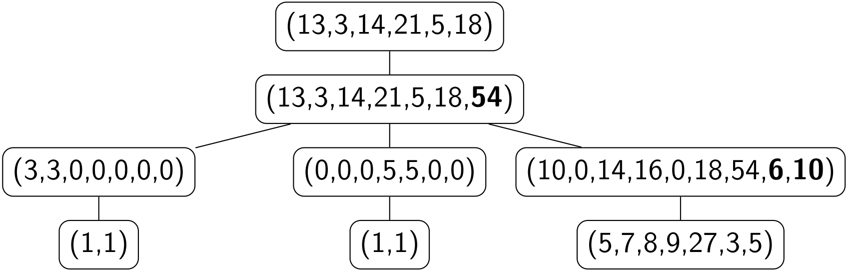

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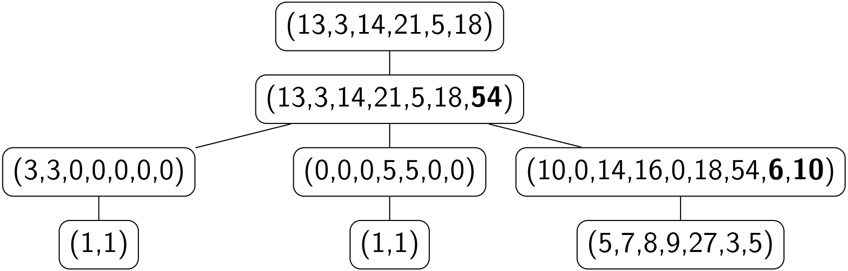
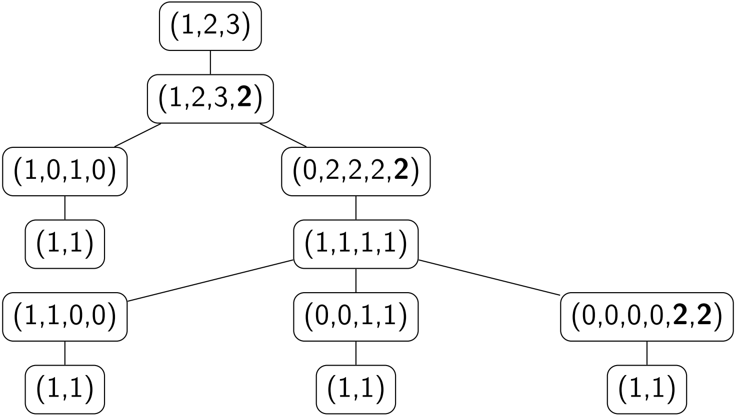


1. Initialize: $2^6 < \mathrm{e}^\mathsf{T}(13,3,14,21,5,18) < 2^7$ with **54** to upper.
2. $\mathrm{e}^\mathsf{T}(13,3,14,21,5,18,54) = 2^7$.
3. Apply split rule to odd power pairs (in this case 2 pairs).
4. $\mathrm{e}^\mathsf{T}(5,7,8,9,27,3,5) = 2^6$.
5. Apply split rule to odd power pairs (in this case 3 pairs).
6. $\mathrm{e}^\mathsf{T}(1,4,9,1,5,9,3) = 2^5 \ldots$

Let's play Tower Tycoon

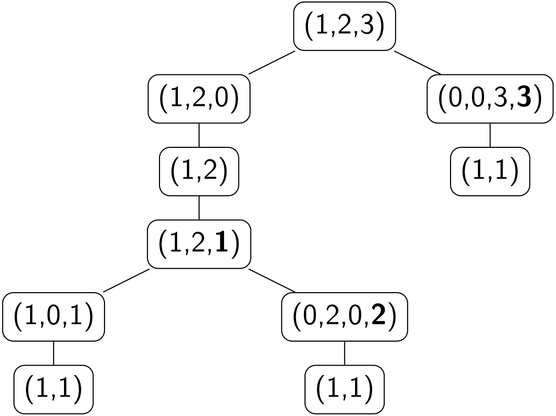
Still room for improvement



$|S| = 4$ if initial cone has a simple base, and $|S| = 5$ otherwise.

Let's play Tower Tycoon

Still room for improvement (subset sum split)



$|S| = 3.$

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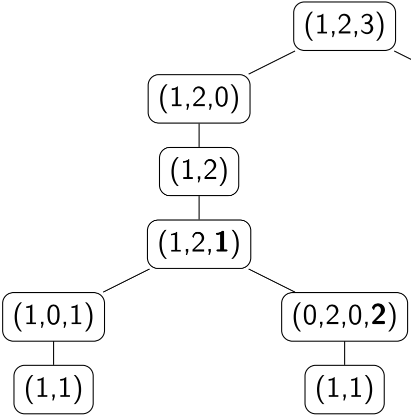

$|S| = 3$. In fact, subset sum splits handle $(1, 2, 3, 6, 12, 24, 48, \ldots)$ in k second-order cones, while the powers of 2 strategy (empirically) uses $2(k-1)$ second-order cones.

Dual information recovery

Split rule

$$\begin{aligned} x^{\alpha} \geq \|z\|_2^{e^T\alpha} \quad &\Leftrightarrow \quad x^{\alpha-\beta}x^{\beta} \geq \|z\|_2^{e^T\alpha}, \\ &\Leftrightarrow \quad x^{\alpha-\beta}u^{e^T\beta} \geq \|z\|_2^{e^T\alpha}, \quad x^{\beta} \geq u^{e^T\beta} \end{aligned}$$

Expansion rule

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Dual information recovery

Split rule

	BEFORE	AFTER
PRIMAL	$\left(\begin{smallmatrix} x \\ z \end{smallmatrix}\right) \in \mathcal{P}^n_\alpha \quad \left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]$	$\begin{pmatrix} x \\ x \\ z \end{pmatrix} \in \mathcal{P}^n_{(\alpha-\beta,\beta)} \quad \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau \end{bmatrix}$
DUAL	$x: \quad +s$ $z: \quad +t$ where $(s,t) \in \left(\mathcal{P}^n_\alpha\right)^*$	$x: \quad +\sigma_1 + \sigma_2$ $z: \quad +\tau$ where $(\sigma_1, \sigma_2, \tau) \in \left(\mathcal{P}^n_{(\alpha-\beta,\beta)}\right)^*$

Recover as $(s,t) \leftarrow (\sigma_1 + \sigma_2, \tau)$.

Dual information recovery

Expansion rule

	BEFORE	AFTER
PRIMAL	$\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathcal{P}_{\alpha}^{n_x+1} \quad \left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]$ $(u \geq 0)$	$\begin{pmatrix} x \\ u \\ u \end{pmatrix} \in \mathcal{P}_{(\alpha,\beta)}^{n_x+2} \quad \begin{bmatrix} \sigma \\ \tau_1 \\ \tau_2 \end{bmatrix}$ $(u \geq 0)$
DUAL	$x: \quad +s$ $u: \quad +t \; (\leq 0)$ where $(s, t) \in \left(\mathcal{P}_{\alpha}^{n_x+1}\right)^*$	$x: \quad +\sigma$ $u: \quad +\tau_1 + \tau_2 \; (\leq 0)$ where $(\sigma, \tau_1, \tau_2) \in \left(\mathcal{P}_{(\alpha,\beta)}^{n_x+2}\right)^*$

Recover as $(s, t) \leftarrow (\sigma, \tau_1 + \tau_2)$.

Dual information recovery

Prerequisites of an elegant proof

Dual split rule

$$\begin{aligned}(x, z) \in (\mathcal{P}_{\alpha}^{n})^* \quad &\Leftrightarrow \quad (x - u, u, z) \in \left(\mathcal{P}_{(\alpha-\beta,\beta)}^{n}\right)^*,\\ &\Leftrightarrow \quad (x - u, v, z) \in \left(\mathcal{P}_{(\alpha-\beta,\mathsf{e}^\mathsf{T}\beta)}^{n}\right)^*, (u, v) \in \left(\mathcal{P}_{\mathsf{e}^\mathsf{T}\beta}^{n}\right)^*.\end{aligned}$$

Dual expansion rule

$$(x, z) \in (\mathcal{P}_{\alpha}^{n_x+1})^*,\ z \geq 0 \quad \Leftrightarrow \quad (x, u, z + u) \in \left(\mathcal{P}_{(\alpha,\beta)}^{n_x+2}\right)^*.$$

The AM-GM inequality does it all:

$$(\mathrm{e}^{\mathsf{T}}\alpha)^{-1}(\alpha^T x) \geq \sqrt[\mathrm{e}^{\mathsf{T}}\alpha]{x^{\alpha}},$$

for $x, \alpha \in \mathbb{R}_+^k$ where $\mathrm{e}^{\mathsf{T}}\alpha > 0$.

Bonus info

It gives rise to a family of outer approximations, the simplest of which is a quadratic cone:

$$\mathcal{P}_\alpha^n \subseteq \{(x, z) \in \mathbb{R}_+^k \times \mathbb{R}^{n-k} \mid (\mathrm{e}^{\mathsf{T}}\alpha)^{-1}(\alpha^T x) \geq \|z\|_2\},$$

Numerical results

Shooting sparrows with a cannon

The 8'th root of 42 is 1.5955343603, but also the infimum of

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & y = 42, \\ & (y, 1, x) \in \mathcal{P}^3_{(1,7)}. \end{array}$$

```
ITE PFEAS    DFEAS    GFEAS    PRSTATUS   POBJ              DOBJ              MU       TIME
0   5.5e+00  1.0e+00  1.0e+00  0.00e+00   0.000000000e+00   0.000000000e+00   1.0e+00  0.01
1   1.1e+00  2.1e-01  1.5e-01  -6.56e-01  2.184849059e-01   -1.207084580e+00  2.1e-01  0.01
2   2.2e-01  3.9e-02  5.4e-02  3.82e-01   5.765513222e-01   1.852211210e-01   3.9e-02  0.01
3   4.2e-02  7.8e-03  2.0e-02  7.43e-01   1.340272353e+00   1.221223568e+00   7.8e-03  0.01
4   7.2e-03  1.3e-03  7.7e-03  8.65e-01   1.539177880e+00   1.515646623e+00   1.3e-03  0.01
5   3.1e-04  5.6e-05  1.6e-03  9.55e-01   1.593269995e+00   1.592202275e+00   5.6e-05  0.01
6   7.0e-06  1.3e-06  2.3e-04  9.98e-01   1.595487015e+00   1.595462738e+00   1.3e-06  0.01
7   2.6e-07  4.8e-08  4.5e-05  1.00e+00   1.595532790e+00   1.595531871e+00   4.8e-08  0.01
8   1.6e-08  2.9e-09  1.1e-05  1.00e+00   1.595534274e+00   1.595534219e+00   2.9e-09  0.01
Optimizer terminated. Time: 0.03

Interior-point solution summary
  Problem status  : PRIMAL_AND_DUAL_FEASIBLE
  Solution status : OPTIMAL
  Primal.  obj: 1.5955342736e+00    nrm: 4e+01    Viol.  con: 9e-09    var: 0e+00    cones: 0e+00
  Dual.    obj: 1.5955342195e+00    nrm: 1e+00    Viol.  con: 0e+00    var: 1e-08    cones: 3e-09
```

Two quadratic cones after presolve. Complementarity is $x^T s = 3.388688\mathrm{e}{-08}$ after dual information recovery.