Solving conic optimization problems using MOSEK

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Section 1

Background

The company

- MOSEK is a Danish company founded 15th of October 1997.
- Vision: Create and sell software for mathematical optimization problems.
  - Linear and conic problems.
  - Convex quadratically constrained problems.
  - Also mixed-integer versions of the above.
- Located in Copenhagen Denmark at Symbion Science Park.
- Daily management: Erling D. Andersen.

Section 2

Outline

Todays topics

- Conic optimization
  - What is it?
  - And why?
  - What can be modelled?
- MOSEK
  - What is it?
  - Examples of usage.
  - Technical details.
- Computational results.

Section 3

Conic optimization

The classical linear optimization problem:

$$\begin{array}{lll} \text{minimize} & c^T x & \\ \text{subject to} & Ax & = \quad b, \\ & x \geq 0. & \end{array}$$

Pro:

- Structure is explicit and simple.
- Data is simple: c, A, b.
- Structure implies convexity i.e. data independent.
- Powerful duality theory including Farkas lemma.
- Smoothness, gradients, Hessians are not an issue.

Therefore, we have powerful algorithms and software.

linear optimization cont.

Con:

- It is linear only.

The classical nonlinear optimization problem

The classical nonlinear optimization problem:

$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to} & g(x) & \leq \quad 0. \end{array}$$

Pro

- It is very general.

Con:

- Structure is hidden.
  - How to specify the problem at all in software?
  - How to compute gradients and Hessians if needed?
  - How to do presolve?
  - How to exploit structure e.g. linearity?
- Convexity checking!
  - Verifying convexity is NP hard.
  - Solution: Disciplined modelling by Grant, Boyd and Ye [6] to assure convexity.

The question

Is there a class of nonlinear optimization problems that preserve almost all of the good properties of the linear optimization problem?

Generic conic optimization problem

Primal form

$$\begin{aligned} \text{minimize} \quad & \sum_k (c^k)^T x^k \\ \text{subject to} \quad & \sum_k A^k x^k = b, \\ & x^k \in \mathcal{K}^k, \qquad \forall k, \end{aligned}$$

where

- $c^k \in \mathbb{R}^{n^k}$,
- $A^k \in \mathbb{R}^{m \times n^k}$,
- $b \in \mathbb{R}^m$,
- $\mathcal{K}^k$ are convex cones.

Convex cone definition

$\mathcal{K}^k$ is a nonempty pointed convex cone i.e.

- (Convexity) $\mathcal{K}$ is a convex set.
- (Conic) $x \in \mathcal{K} \Rightarrow \lambda x \in \mathcal{K},\ \forall \lambda \geq 0.$
- (Pointed) $x \in \mathcal{K}$ and $-x \in \mathcal{K} \Rightarrow x = 0.$

Comments:

- Wikipedia reference:
  `https://en.wikipedia.org/wiki/Convex_cone`.

An alternative conic model

Dual form

$$\begin{array}{lll} \text{maximize} & b^T y & \\ \text{subject to} & c^k - (A^k)^T y & \in \quad (\mathcal{K}^k)^*, \forall k. \end{array}$$

- $(\mathcal{K}^k)^*$ corresponding dual cones.
- Equally general.
- Problems are convex.
- The objective sense is not important.

Beauty of conic optimization

- Separation of data and structure:
  - Data: c^k, A^k and b.
  - Structure: $\mathcal{K}$.
- See Nemirovski [10] for more details.
- Cheating: f has just been replaced by $\mathcal{K}$ or?

The 5 cones

Almost all practical convex optimization models can be formulated with **5** convex cone types:

- Linear.
- Quadratic.
- Semidefinite.
- Exponential.
- Power.

Evidence:

- Lubin [8] shows all convex instances(85) in the benchmark library MINLPLIB2 is conic representable using the 5 cones.

Extremely disciplined optimization

Further comments:

- The cones have an explicit structure as shown subsequently.
- Leads to structural convexity.
- Extremely disciplined modelling!

So **NO** cheating.

The cones

The linear cone:

$$\{x \in \mathbb{R} : \ x \geq 0\}.$$

The quadratic cones:

$$\begin{aligned} \mathcal{K}_q &:= \left\{ x \in \mathbb{R}^n : \ x_1 \geq \sqrt{\sum_{j=2}^{n} x_j^2} \right\}, \\ \mathcal{K}_r &:= \left\{ x \in \mathbb{R}^n : \ 2x_1 x_2 \geq \sum_{j=3}^{n} x_j^2, \ x_1, x_2 \geq 0 \right\}. \end{aligned}$$

Examples:

$$
\begin{aligned}
(t,x) \in \mathcal{K}_q &\Leftrightarrow t \geq \|x\|\,, \\
(2,t,V^Tx) \in \mathcal{K}_r &\Leftrightarrow t \geq x^TVV^Tx, \\
\left\{(t,x) \mid t \geq \frac{1}{x},\, x \geq 0\right\} &\Leftrightarrow \left\{(t,x) \mid (x,t,\sqrt{2}) \in \mathcal{K}_r^3\right\}, \\
\left\{(t,x) \mid t \geq x^{3/2},\, x \geq 0\right\} &\Leftrightarrow \left\{(t,x) \mid (s,t,x),(x,1/8,s) \in \mathcal{K}_r^3\right\}, \\
\left\{(t,x) \mid t \geq \frac{1}{x^2},\, x \geq 0\right\} &\Leftrightarrow \left\{(t,x) \mid (t,1/2,s),(x,s,\sqrt{2}) \in \mathcal{K}_r^3\right\},
\end{aligned}
$$

More examples:

- $$\left\{(t,x) \mid t \geq x^{5/3},\, x \geq 0\right\}$$
$$\Leftrightarrow$$
$$\left\{(t,x) \mid (s,t,x),(1/8,z,s),(s,x,z) \in \mathcal{K}_r^3\right\},$$

- $$\left\{(t,x,y) \mid t \geq \frac{|x|^3}{y^2}, y \geq 0\right\}$$
$$\Leftrightarrow$$
$$\left\{(t,x,y) \mid (z,x) \in \mathcal{K}_q^2, (\frac{y}{2},s,z), (\frac{t}{2},z,s) \in \mathcal{K}_r^3\right\}.$$

- A rational function:

$$t \geq x^p,\, x \geq 0,\, p \geq 1 \text{ is rational}$$

is conic quadratic representable.

The cones: continued

The cone of symmetric positive semidefinite matrices:

$$\mathcal{K}_s \quad := \quad \{X \in \mathbb{R}^{n \times n} \mid \quad X = X^T, \quad \lambda_{\min}(X) \geq 0\}$$

Examples:

- Approximation of nonconvex problems e.g graph partitioning.
- Nearest correlation matrix.

Symmetric cones

Facts:

- The 3 cones belong to the class of symmetric cones.
  - Are **SELF-DUAL**.
  - Are homogeneous.
- Only 5 different symmetric cones.

The power cone

The power cone:

$$\mathcal{K}_{pow}(\alpha) := \left\{ (x,z) : \prod_{j=1}^{n} x_j^{|\alpha_j|} \geq \|z\|^{\sum_{j=1}^{n}|\alpha_j|},\ x \geq 0 \right\}.$$

Examples ($\alpha \in (0,1)$):

$$\begin{aligned}
(t,1,x) \in \mathcal{K}_{pow}(\alpha, 1-\alpha) &\Leftrightarrow t \geq |x|^{1/\alpha},\ t \geq 0,\\
(x,1,t) \in \mathcal{K}_{pow}(\alpha, 1-\alpha) &\Leftrightarrow x^{\alpha} \geq |t|,\ x \geq 0,\\
(x,t) \in \mathcal{K}_{pow}(e) &\Leftrightarrow \left(\prod_{j=1}^{n} x_j\right)^{1/n} \geq |t|,\ x \geq 0.
\end{aligned}$$

More examples from Chares [4]:

- p-norm:

$$t \geq \|x\|_p .$$

- l_p cone:

$$\left\{ (x,t,s) : \sum_{j=1}^{n} \left(\frac{1}{p_i} \left(\frac{|x_j|}{t} \right)^{p_j} \right) \leq \frac{s}{t},\, t \geq 0 \right\}$$

  where $p > 0$.

Dual power cone

- Is self-dual using a redefined inner-product.

The exponential cone

The exponential cone

$$\begin{aligned}\mathcal{K}_{\exp} \quad := \quad & \{(x_1,x_2,x_3):\ x_1 \geq x_2 e^{\frac{x_3}{x_2}},\ x_2 \geq 0\} \\ & \cup\{(x_1,x_2,x_3):\ x_1 \geq 0, x_2 = 0,\ x_3 \leq 0\}\end{aligned}$$

Applications:

$$\begin{aligned}(t,1,x) \in \mathcal{K}_{\exp} \quad &\Leftrightarrow \quad t \geq e^x \\ (x,1,t) \in \mathcal{K}_{\exp} \quad &\Leftrightarrow \quad t \leq \ln(x), \\ (1,x,t) \in \mathcal{K}_{\exp} \quad &\Leftrightarrow \quad t \leq -x\ln(x), \\ (y,x,-t) \in \mathcal{K}_{\exp} \quad &\Leftrightarrow \quad t \geq x\ln(x/y), \text{(relative entropy)}.\end{aligned}$$

More examples:

- Geometric programming (Duffin, Kortanek, Peterson, ...)
- Lambert function [4, 3].

Dual exponential cone

$$
\begin{aligned}
(\mathcal{K}_{\exp})^* \quad := \quad & \{(s_1, s_2, s_3) : \; s_1 \geq (-s_3) e^{-1} e^{\frac{s_3}{s_2}}, \, s_3 \leq 0\} \\
& \cup \{(s_1, s_2, s_3) : \; s_1 \geq 0, s_2 \geq 0, \, s_3 = 0\}
\end{aligned}
$$

Summary

- Conic modelling using the linear, quadratic, semidefinite, power, and exponential cones. (More cones in the future?).
- Inspired by disciplined modeling of Grant, Boyd and Ye [6].
- Models can be solved efficiently using a primal-dual interior-point method.
- Preserves all the good properties of linear optimization:
  - Simple and explicit data.
  - Structural convexity.
  - Duality (**almost**).
  - No issues with smoothness and differentiability.

Section 4

MOSEK

Mosek

- A software package.
- Solves large-scale sparse optimization problems.
- Handles linear and **conic** problems.
- Stand-alone as well as embedded.
- Version 1 released in 1999.
- Version 8 to be released Fall 2016.
- Version 9 to be released in 2018.
    - Includes support for the 2 nonsymmetric cones.

For details about interfaces, trials, academic license etc. see

`https://mosek.com`.

Illustrative usage

- Fusion interface for Python.
  - Java, .NET and C++ avilability.
- Matlab toolbox.
  - Also avilable for R.

An investor can invest in n stocks or assets to be held over a period of time. What is the optimal portfolio?
Now assume a stochastic model where the return of the assets is a random variable

$$r$$

with known mean

$$\mu = \mathbf{E}r$$

and covariance

$$\Sigma = \mathbf{E}(r - \mu)(r - \mu)^T.$$

Return and risk

Let x_j be the amount invested in asset j. Moreover, the expected return is:

$$\mathbf{E}y = \mu^T x$$

and variance of the return is

$$(y - \mathbf{E}y)^2 = x^T \Sigma x.$$

The investors optimization problem:

$$\begin{array}{lccc} \text{maximize} & \mu^T x & & \\ \text{subject to} & e^T x & = & w + e^T x^0, \\ & x^T \Sigma x & \leq & \gamma^2, \\ & x & \geq 0, & \end{array}$$

where

- e is the vector of all ones.
- w investors initial wealth.
- x^0 investors initial portfolio.
- Objective maximize expected return.
- Constraints:
  - Budget constraint. $(e^T x = \sum_{j=1}^{n} x_j)$.
  - Risk constraint. γ is chosen by the investor.
  - Only buy a positive amount i.e. no short-selling.

The covariance matrix Σ is positive semidefinite by definition. Therefore,

$$\exists G: \quad \Sigma = GG^T.$$

CQ reformulation:

$$\begin{array}{lccl} \text{maximize} & \mu^T x & & \\ \text{subject to} & e^T x & = & w + e^T x^0, \\ & [\gamma; G^T x] & \in & \mathcal{K}_q^{n+1}, \\ & x & \geq & 0. \end{array}$$

because

$$[\gamma; G^T x] \in \mathcal{K}_q^{n+1} \Rightarrow \gamma \geq \left\| G^T x \right\| \Rightarrow \gamma^2 \geq x^T G G^T x.$$

Fusion for python program

portfolio_basic.py

```python
import mosek
import sys

from mosek.fusion   import *
from portfolio_data import *

def BasicMarkowitz(n,mu,GT,x0,w,gamma):
    with  Model("Basic Markowitz") as M:

        # Redirect log output from the solver to stdout for debugging.
        # if uncommented.
        M.setLogHandler(sys.stdout)

        # Defines the variables (holdings). Shortselling is not allowed.
        x = M.variable("x", n, Domain.greaterThan(0.0))

        #  Maximize expected return
        M.objective('obj', ObjectiveSense.Maximize, Expr.dot(mu,x))

        # The amount invested  must be identical to initial wealth
        M.constraint('budget', Expr.sum(x), Domain.equalsTo(w+sum(x0)))

        # Imposes a bound on the risk
        M.constraint('risk', Expr.vstack( gamma,Expr.mul(GT,x)), Domain.inQCone())

        M.solve()

        return (M.primalObjValue(), x.level())

if __name__ == '__main__':

    (expret,x) = BasicMarkowitz(n,mu,GT,x0,w,gamma)
    print("Expected return: %e" % expret)
    print("x: "),
    print(x)
```

Python data

portfolio_data.py

```
n     = 3;
w     = 1.0;
mu    = [0.1073,0.0737,0.0627]
x0    = [0.0,0.0,0.0]
gamma = 0.04
GT    = [[ 0.166673333200005, 0.0232190712557243 ,  0.0012599496030238 ],
          [ 0.0              , 0.102863378954911  , -0.00222873156550421],
          [ 0.0              , 0.0                ,  0.0338148677744977 ]]
```

Running

Running

```
python portfolio_basic.py
```

Output

MOSEK optimizer log

```
Optimizer  - threads                : 4
Optimizer  - solved problem         : the primal
Optimizer  - Constraints            : 3
Optimizer  - Cones                  : 1
Optimizer  - Scalar variables       : 6                 conic                  : 4
Optimizer  - Semi-definite variables: 0                 scalarized             : 0
Factor     - setup time             : 0.00              dense det. time        : 0.00
Factor     - ML order time          : 0.00              GP order time          : 0.00
Factor     - nonzeros before factor : 6                 after factor           : 6
Factor     - dense dim.             : 0                 flops                  : 7.00e+001
ITE PFEAS    DFEAS    GFEAS    PRSTATUS   POBJ              DOBJ              MU       TIME
0   1.0e+000 1.0e+000 1.0e+000 0.00e+000  0.000000000e+000  0.000000000e+000  1.0e+000 0.00
1   1.7e-001 1.7e-001 4.4e-001 9.46e-001  1.259822223e-001  2.171837612e-001  1.7e-001 0.00
2   4.0e-002 4.0e-002 5.6e-001 1.56e+000  8.104070951e-002  1.693911786e-001  4.0e-002 0.00
3   1.4e-002 1.4e-002 2.9e-001 3.00e+000  7.268285567e-002  8.146211968e-002  1.4e-002 0.00
4   1.3e-003 1.3e-003 1.1e-001 1.43e+000  7.102726686e-002  7.178857777e-002  1.3e-003 0.00
5   1.7e-004 1.7e-004 3.9e-002 1.05e+000  7.101472221e-002  7.111329525e-002  1.7e-004 0.00
6   7.7e-006 7.7e-006 8.5e-003 1.01e+000  7.099770619e-002  7.100232290e-002  7.7e-006 0.00
7   6.0e-007 6.0e-007 2.4e-003 1.00e+000  7.099794084e-002  7.099830405e-002  6.0e-007 0.00
8   1.7e-008 1.7e-008 4.0e-004 1.00e+000  7.099799652e-002  7.099800667e-002  1.7e-008 0.00
Interior-point optimizer terminated. Time: 0.00.

Optimizer terminated. Time: 0.01
Expected return: 7.099800e-02
x:
[ 0.15518625  0.12515363  0.71966011]
```

The efficient frontier

- Question: What is the right γ?
- Answer: Compute the efficient frontier i.e. optimal combinations of expected return and risk.

I.e. solve

$$\begin{array}{lll} \text{maximize} & \mu^T x - \alpha s & \\ \text{subject to} & e^T x & = w + e^T x^0, \\ & [s; G^T x] & \in \mathcal{K}_q^{n+1}, \\ & x & \geq 0 \end{array}$$

for all $\alpha \in [0, \infty[$.

Python program

portfolio_frontier.py

```
import mosek
import sys

from mosek.fusion   import *
from portfolio_data import *

def EfficientFrontier(n,mu,GT,x0,w,alphas):
    with Model("Efficient frontier") as M:
        # Defines the variables (holdings). Shortselling is not allowed.
        x = M.variable("x", n, Domain.greaterThan(0.0)) # Portfolio variables
        s = M.variable("s", 1, Domain.unbounded()) # Risk variable

        M.constraint('budget', Expr.sum(x), Domain.equalsTo(w+sum(x0)))

        # Computes the risk
        M.constraint('risk', Expr.vstack(s,Expr.mul(GT,x)),Domain.inQCone())

        frontier = []

        mudotx   = Expr.dot(mu,x) # Is reused.

        for i,alpha in enumerate(alphas):

            #  Define objective as a weighted combination of return and risk
            M.objective('obj', ObjectiveSense.Maximize, Expr.sub(mudotx,Expr.mul(alpha,s)))

            M.solve()

            frontier.append((alpha,M.primalObjValue(),s.level()[0]))

        return frontier

if __name__ == '__main__':
    alphas   = [x * 0.1 for x in range(0, 21)]
    frontier = EfficientFrontier(n,mu,GT,x0,w,alphas)
    print('%-14s %-14s %-14s %-14s' % ('alpha','obj','exp. ret', 'std. dev.'))
    for f in frontier:
        print("%-14.2e %-14.2e %-14.2e %-14.2e" % (f[0],f[1],f[1]+f[0]*f[2],f[2])),
```

Observations

- The whole model is not rebuild for each α.
- Leads to better efficiency.

Output

alpha	obj	exp. ret	std. dev.
0.00e+00	1.07e-01	1.07e-01	1.67e-01
1.00e-01	9.06e-02	1.07e-01	1.67e-01
2.00e-01	7.40e-02	1.07e-01	1.67e-01
3.00e-01	6.01e-02	8.05e-02	6.81e-02
4.00e-01	5.50e-02	7.20e-02	4.23e-02
5.00e-01	5.11e-02	6.98e-02	3.73e-02
6.00e-01	4.75e-02	6.86e-02	3.53e-02
7.00e-01	4.40e-02	6.79e-02	3.42e-02
8.00e-01	4.06e-02	6.74e-02	3.35e-02
9.00e-01	3.73e-02	6.71e-02	3.31e-02
1.00e+00	3.40e-02	6.68e-02	3.28e-02
1.10e+00	3.07e-02	6.66e-02	3.26e-02
1.20e+00	2.75e-02	6.64e-02	3.24e-02
1.30e+00	2.42e-02	6.62e-02	3.23e-02
1.40e+00	2.10e-02	6.61e-02	3.22e-02
1.50e+00	1.78e-02	6.60e-02	3.21e-02
1.60e+00	1.46e-02	6.59e-02	3.21e-02
1.70e+00	1.14e-02	6.58e-02	3.20e-02
1.80e+00	8.19e-03	6.57e-02	3.20e-02
1.90e+00	5.00e-03	6.57e-02	3.19e-02
2.00e+00	1.81e-03	6.56e-02	3.19e-02

Alternative model

Formulated with variance:

$$\begin{array}{lccl} \text{maximize} & \mu^T x - \alpha s & & \\ \text{subject to} & e^T x & = & w + e^T x^0, \\ & [0.5; s; G^T x] & \in & \mathcal{K}_r^{n+1}, \\ & x & \geq & 0 \end{array}$$

implying

$$s \geq x^T G G^T x.$$

which is the traditional Markowitz model.

Variance or square root of variance?

- Expected return and $\sqrt{V(x)}$ has same unit i.e. USD.
- Makes choice of α easier.
- $\sqrt{V(x)}$ leads to better model scaling.

Market impact costs

- Question: Is prices on asserts independent of trade volume.
- Answer: No. Why?

A common assumption about market impact costs are

$$m_j \sqrt{|x_j - x_j^0|}$$

where $m_j \geq 0$ is a constant that is estimated in some way [7][p. 452].
Therefore,

$$T_j(x_j - x_j^0) = m_j |x_j - x_j^0| \sqrt{|x_j - x_j^0|} = m_j |x_j - x_j^0|^{3/2}$$

can be seen as a transaction cost.

Formulated using a power cone

upcoming version 9 code

Clearly

$$[t_j; 1; x_j - x_j^0] \in \mathcal{K}_{pow}\left([2; 1]\right)$$

implies

$$t_j^2 1 \geq |x_j - x_j^0|^3$$

and hence

$$t_j \geq |x_j - x_j^0|^{3/2}.$$

Revised model with market impact costs

The model:

$$\begin{array}{lllll} \text{maximize} & \mu^T x & & & \\ \text{subject to} & e^T x + m^T t & = & w + e^T x^0, & \\ & [\gamma; G^T x] & \in & \mathcal{K}_q^{n+1}, & \\ & [t_j; 1; x_j - x_j^0] & \in & \mathcal{K}_{pow}\left([2;1]\right), & j = 1, \ldots, n, \\ & x & \geq & 0. & \end{array}$$

The revised budget constraint is

$$e^T x = w + e^T x^0 - m^T t$$

where

$$m^T t$$

is the total market impact cost that must be paid too.

Python program

portfolio_marketimpact.py

```
import mosek
import numpy
import sys

from mosek.fusion   import *
from portfolio_data import *

def MarkowitzWithMarketImpact(n,mu,GT,x0,w,gamma,m):
    with  Model("Markowitz portfolio with market impact") as M:

        M.setLogHandler(sys.stdout)

        # Defines the variables. No shortselling is allowed.
        x = M.variable("x", n, Domain.greaterThan(0.0))

        # Additional "helper" variables
        t = M.variable("t", n, Domain.unbounded())

        #  Maximize expected return
        M.objective('obj', ObjectiveSense.Maximize, Expr.dot(mu,x))

        # Invested amount + slippage cost = initial wealth
        M.constraint('budget', Expr.add(Expr.sum(x),Expr.dot(m,t)), Domain.equalsTo(w+sum(x0)))

        # Imposes a bound on the risk
        M.constraint('risk', Expr.vstack(gamma,Expr.mul(GT,x)), Domain.inQCone())

        # (t_j^2/3*1^1/3) >= |x_j -x0_j|
        M.constraint('t', Expr.hstack(t,Expr.constTerm(n,1.0),Expr.sub(x,x0)),Domain.inPPowerCone(2.0/3.0))

        M.solve()

        print('Expected return: %.4e Std. deviation: %.4e Market impact cost: %.4e' % \
              (M.primalObjValue(),gamma,numpy.dot(m,t.level())))

if __name__ == '__main__':
    m = n*[1.0e-2]
    MarkowitzWithMarketImpact(n,mu,GT,x0,w,gamma,m)
```

Now

```
M.constraint('t',
             Expr.hstack(t,
                         Expr.constTerm(n,1.0),
                         Expr.sub(x,x0)),
             Domain.inPPowerCone(2.0/3.0))
```

implies each row of

$$\begin{bmatrix} t_0 & 1 & x_0 - x_0^0 \\ t_1 & 1 & x_1 - x_1^0 \\ \vdots & \vdots & \vdots \\ t_{n-1} & 1 & x_{n-1} - x_{n-1}^0 \end{bmatrix}$$

belong to the power cone.

Output

```
ITE PFEAS    DFEAS    GFEAS    PRSTATUS   POBJ              DOBJ              MU       TIME
0   1.0e+00  1.3e+00  1.0e+00  0.00e+00   0.000000000e+00   0.000000000e+00   1.0e+00  0.00
1   2.1e-01  2.7e-01  7.7e-01  2.19e+00   7.318882175e-02   2.266685099e-01   2.2e-01  0.01
2   5.0e-02  6.4e-02  4.0e-01  1.37e+00   7.876398241e-02   1.086059026e-01   5.1e-02  0.01
3   1.4e-02  1.8e-02  2.1e-01  1.27e+00   7.092889623e-02   7.711273002e-02   1.4e-02  0.01
4   2.7e-03  3.5e-03  9.1e-02  1.31e+00   6.913397279e-02   6.979818332e-02   2.6e-03  0.01
5   5.5e-04  7.1e-04  5.2e-02  1.28e+00   6.976332845e-02   7.004123863e-02   5.6e-04  0.01
6   1.1e-04  1.4e-04  4.4e-02  1.47e+00   7.046711382e-02   7.054715754e-02   1.2e-04  0.01
7   7.9e-05  1.0e-04  3.9e-02  1.14e+00   7.052317150e-02   7.057952499e-02   9.0e-05  0.01
8   3.1e-05  4.0e-05  2.5e-02  1.07e+00   7.057379270e-02   7.059477263e-02   3.7e-05  0.01
9   3.7e-06  4.8e-06  8.9e-03  1.03e+00   7.061519841e-02   7.061767895e-02   4.4e-06  0.01
10  1.5e-06  2.0e-06  5.8e-03  1.00e+00   7.061761643e-02   7.061862173e-02   1.8e-06  0.01
11  3.3e-07  4.2e-07  2.7e-03  1.00e+00   7.061846400e-02   7.061867536e-02   4.1e-07  0.01
12  4.5e-08  5.8e-08  1.0e-03  1.00e+00   7.061878727e-02   7.061881600e-02   5.6e-08  0.01
Optimizer terminated. Time: 0.01

Expected return: 7.0619e-02 Std. deviation: 4.0000e-02 Market impact cost: 7.0497e-03
```

Formulated using quadratic cones

Recall from earlier

$$\{(t,z) : t \geq z^{3/2}, z \geq 0\} = \{(t,z) : (s,t,z), (z,1/8,s) \in \mathcal{K}_r^3\}.$$

So

$$\begin{array}{ll} z_j & = |x_j - x_j^0|, \\ (s_j, t_j, z_j), (z_j, 1/8, s_j) & \in \mathcal{K}_r^3, \\ \sum_{j=1}^{n} T(x_j - x_j^0) & = \sum_{j=1}^{n} t_j. \end{array}$$

and the relaxation

$$\begin{array}{ll} z_j & \geq |x_j - x_j^0|, \\ (s_j, t_j, z_j), (z_j, 1/8, s_j) & \in \mathcal{K}_r^3, \\ \sum_{j=1}^{n} T(x_j - x_j^0) & = \sum_{j=1}^{n} t_j \end{array}$$

is good enough.
Now

$$z_j \geq |x_j - x_j^0| \Leftrightarrow [z_j; x_j - x_j^0] \in \mathcal{K}_q^2.$$

Revised model with market impact costs

$$
\begin{array}{lllll}
\text{maximize} & \mu^T x & & & \\
\text{subject to} & e^T x + m^T t & = & w + e^T x^0, & \\
& [\gamma; G^T x] & \in & \mathcal{K}_q^{n+1}, & \\
& [z_j; x_j - x_j^0] & \in & \mathcal{K}_q^2, & j = 1, \ldots, n, \\
& [v_j; t_j; z_j], [z_j; 1/8; v_j] & \in & \mathcal{K}_r^3, & j = 1, \ldots, n, \\
& x & \geq & 0. &
\end{array}
$$

Python program

portfolio_marketimpact.py

```
import mosek
import numpy
import sys

from mosek.fusion   import *
from portfolio_data import *

def MarkowitzWithMarketImpact(n,mu,GT,x0,w,gamma,m):
    with  Model("Markowitz portfolio with market impact") as M:

        M.setLogHandler(sys.stdout)

        # Defines the variables. No shortselling is allowed.
        x = M.variable("x", n, Domain.greaterThan(0.0))

        # Additional "helper" variables
        t = M.variable("t", n, Domain.unbounded())
        z = M.variable("z", n, Domain.unbounded())
        v = M.variable("v", n, Domain.unbounded())

        #  Maximize expected return
        M.objective('obj', ObjectiveSense.Maximize, Expr.dot(mu,x))

        # Invested amount + slippage cost = initial wealth
        M.constraint('budget', Expr.add(Expr.sum(x),Expr.dot(m,t)), Domain.equalsTo(w+sum(x0)))

        # Imposes a bound on the risk
        M.constraint('risk', Expr.vstack(gamma,Expr.mul(GT,x)), Domain.inQCone())

        # z >= |x-x0| componentwise
        M.constraint('trade', Expr.hstack(z,Expr.sub(x,x0)), Domain.inQCone())

        # t >= z^1.5, z >= 0.0. Needs two rotated quadratic cones to model this term
        M.constraint('ta', Expr.hstack(v,t,z),Domain.inRotatedQCone())
        M.constraint('tb', Expr.hstack(z,Expr.constTerm(n,1.0/8.0),v),Domain.inRotatedQCone())

        M.solve()

        print('Expected return: %.4e Std. deviation: %.4e Market impact cost: %.4e' % \
              (M.primalObjValue(),gamma,numpy.dot(m,t.level())))

if __name__ == '__main__':
    m = n*[1.0e-2]
    MarkowitzWithMarketImpact(n,mu,GT,x0,w,gamma,m)
```

Output

```
ITE PFEAS    DFEAS    GFEAS    PRSTATUS   POBJ              DOBJ              MU       TIME
0   1.0e+00  1.0e+00  1.0e+00  0.00e+00   0.000000000e+00   0.000000000e+00   1.0e+00  0.00
1   1.8e-01  1.8e-01  9.5e-01  1.48e+00   7.747680092e-02   4.556587652e-01   1.8e-01  0.01
2   5.3e-02  5.3e-02  5.1e-01  2.31e+00   7.935900070e-02   1.273387740e-01   5.3e-02  0.01
3   2.0e-02  2.0e-02  3.7e-01  1.35e+00   7.108283206e-02   8.824719666e-02   2.0e-02  0.01
4   2.4e-03  2.4e-03  1.8e-01  1.32e+00   6.919377667e-02   7.133079447e-02   2.4e-03  0.03
5   9.5e-04  9.5e-04  1.2e-01  1.27e+00   7.010222722e-02   7.087227685e-02   9.5e-04  0.03
6   2.1e-04  2.1e-04  7.0e-02  1.29e+00   7.045286219e-02   7.060641849e-02   2.1e-04  0.03
7   3.2e-05  3.2e-05  2.8e-02  1.14e+00   7.059334870e-02   7.061493273e-02   3.2e-05  0.03
8   4.2e-06  4.2e-06  1.0e-02  1.02e+00   7.061524315e-02   7.061809740e-02   4.2e-06  0.03
9   9.5e-08  9.5e-08  1.5e-03  1.00e+00   7.061875274e-02   7.061881641e-02   9.5e-08  0.03
Optimizer terminated. Time: 0.03

Expected return: 7.0619e-02 Std. deviation: 4.0000e-02 Market impact cost: 7.0514e-03
```

Now assume there is a cost associated with trading asset j and the cost is given by

$$T_j(\Delta x_j) = \begin{cases} 0, & \Delta x_j = 0, \\ f_j + g_j|\Delta x_j|, & \text{otherwise}, \end{cases}$$

where

$$\Delta x_j = x_j - x_j^0.$$

Assume U_j is known such that

$$x_j \leq U_j.$$

New model

With transactions costs

$$
\begin{array}{lllll}
\text{maximize} & \mu^T x & & & \\
\text{subject to} & e^T x + \sum_{j=1}^{n} (f_j y_j + g_j z_j) & = & w + e^T x^0, & \\
& [\gamma; G^T x] & \in & \mathcal{K}_q^{n+1} & \\
& [z_j; x_j - x_j^0] & \in & \mathcal{K}_q^2, & j = 1, \ldots, n, \\
& z_j & \leq & U_j y_j, & j = 1, \ldots, n, \\
& y_j & \in & \{0, 1\}, & j = 1, \ldots, n, \\
& x & \geq & 0. &
\end{array}
$$

Observe

- Is a mixed-integer model.
- y_j models whether x_j is traded or not.
- We have $z_j \geq 0$ and hence $y_j = 0 \Rightarrow z_j = 0 \Rightarrow x_j = x_j^0$.
- Choice of U_j is important for computational efficiency. The smaller the better.

Python program

portfolio_transaction.py

```
import mosek
import numpy
import sys

from mosek.fusion   import *
from portfolio_data import *

def MarkowitzWithTransactionsCost(n,mu,GT,x0,w,gamma,f,g):
    # Upper bound on the traded amount
    w0 = w+sum(x0)
    u = n*[w0]

    with Model("Markowitz portfolio with transaction costs") as M:
        x = M.variable("x", n, Domain.greaterThan(0.0))
        z = M.variable("z", n, Domain.unbounded())
        y = M.variable("y", n, Domain.binary())

        #  Maximize expected return
        M.objective('obj', ObjectiveSense.Maximize, Expr.dot(mu,x))

        # Invest amount + transactions costs = initial wealth
        M.constraint('budget', Expr.add([ Expr.sum(x), Expr.dot(f,y),Expr.dot(g,z)] ), Domain.equalsTo(w0))

        # Imposes a bound on the risk
        M.constraint('risk', Expr.vstack( gamma,Expr.mul(GT,x)), Domain.inQCone())

        # z >= |x-x0|
        M.constraint('trade', Expr.hstack(z,Expr.sub(x,x0)),Domain.inQCone())

        # Constraints for turning y off and on. z-diag(u)*y<=0 i.e. z_j <= u_j*y_j
        M.constraint('y_on_off', Expr.sub(z,Expr.mulElm(u,y)), Domain.lessThan(0.0))

        # Integer optimization problems can be very hard to solve so limiting the
        # maximum amount of time is a valuable safe guard
        M.setSolverParam('mioMaxTime', 180.0)
        M.solve()

        print('Expected return: %.4e Std. deviation: %.4e Transactions cost: %.4e' % \
              (numpy.dot(mu,x.level()),gamma,numpy.dot(f,y.level())+numpy.dot(g,z.level())))
        print(x.level())
        print(y.level())


if __name__ == '__main__':
    f = n*[0.1]
    g = n*[0.01]
    MarkowitzWithTransactionsCost(n,mu,GT,x0,w,gamma,f,g)
```

Result:

```
Expected return: 5.8743e-02
Std. deviation: 4.0000e-02
Transactions cost: 2.0792e-01
[ 0.20358627  0.          0.5884929 ]
[ 1.  0.  1.]
```

(Large transactions cost?)

Max likelihood estimator of a convex density function

Estimate of an unknown convex density function

$$g : \mathbb{R}_+ \to \mathbb{R}_+.$$

- Let Y be the real-valued random variable with density function g.
- Let $y_1, ..., y_n$ be an ordered sample of n outcomes of Y.
- Assume $y_1 < y_2 < \ldots < y_n$.
- The estimator of $\tilde{g} \geq 0$ is a piecewise linear function

$$\tilde{g} : [y_1, y_n] \to \mathbb{R}_+$$

with break points at $(y_i, \tilde{g}(yi)), i = 1, \ldots, n$.

See [15] for details.

Let

$$x_i > 0, i = 1, \ldots, n,$$

be the estimator of $g(y_i)$.
The slope for ith segment is given by

$$\frac{x_{i+1} - x_i}{y_{i+1} - y_i}.$$

Hence the convexity requirement is

$$\frac{x_{i+1} - x_i}{y_{i+1} - y_i} \leq \frac{x_{i+2} - x_{i+1}}{y_{i+2} - y_{i+1}}, \forall i = 1, \ldots, n-2.$$

Recall the area under the density function must be 1. Hence,

$$\sum_{i=1}^{n-1}(y_{i+1}-y_i)\left(\frac{x_{i+1}+x_i}{2}\right)=1$$

must hold.
The problem

$$\begin{aligned}
\text{maximize} \quad & \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \\
\text{subject to} \quad & \frac{x_{i+1}-x_i}{y_{i+1}-y_i}-\frac{x_{i+2}-x_{i+1}}{y_{i+2}-y_{i+1}} \leq 0, \quad \forall i=1,\ldots,n-2, \\
& \sum_{i=1}^{n-1}(y_{i+1}-y_i)\left(\frac{x_{i+1}+x_i}{2}\right) = 1, \\
& x \geq 0.
\end{aligned}$$

The power cone variant

Conic reformulation using a power cone:

$$
\begin{array}{lrcll}
\text{maximize} & t & & & \\
\text{subject to} & -\Delta y_{i+1} x_i & & & \\
& +(\Delta y_i + \Delta y_{i+1}) x_{i+1} & & & \\
& -\Delta y_i x_{i+2} & \leq & 0 & \forall i = 1, \ldots, n-2, \\
& \displaystyle\sum_{i=1}^{n-1} \Delta y_i \left(\frac{x_{i+1} + x_i}{2} \right) & = & 1, & \\
& [x; t] \in \mathcal{K}_{pow}(e), & & & \\
& x \geq 0 & & &
\end{array}
$$

where

$$\Delta y_i = y_{i+1} - y_i.$$

The exponential cone variant

Maximizing logarithm of the geometric mean:

$$
\begin{array}{lrcl}
\text{maximize} & \frac{1}{n}\sum_{i=1}^{n} t_i & & \\
\text{subject to} & -\Delta y_{i+1} x_i & & \\
& +(\Delta y_i + \Delta y_{i+1}) x_{i+1} & & \\
& -\Delta y_i x_{i+2} & \leq & 0, \quad \forall i = 1, \ldots, n-2, \\
& \sum_{i=1}^{n-1} \Delta y_i \left(\frac{x_{i+1} + x_i}{2} \right) & = & 1, \\
& [x_i; 1; t_i] \in \mathcal{K}_{\exp}, & & \\
& x \geq 0 & &
\end{array}
$$

where

$$\Delta y_i = y_{i+1} - y_i.$$

Maximum likelyhood density

maxlikeden.py

```
import mosek
import sys

from mosek.fusion import *

def  buildandsolve(name,y):    # y[i+1]-y[i]>0
    with Model("Max likelihood") as M:

        M.setLogHandler(sys.stdout)  # Make sure we get some output

        n      = len(y)

        t      = M.variable('t', n, Domain.unbounded())
        x      = M.variable('x', n, Domain.greaterThan(0.0))

        dy     = [y[i+1]-y[i] for i in range(0,n-1)]

        eleft  = Expr.mulElm(dy[1:n-1],x.slice(0,n-2))
        emid   = Expr.add(Expr.mulElm(dy[0:n-2],x.slice(1,n-1)),Expr.mulElm(dy[1:n-1],x.slice(1,n-1)))
        eright = Expr.mulElm(dy[0:n-2],x.slice(2,n))

        M.constraint('t<=ln(x)',Expr.hstack(x,Expr.constTerm(n,1.0),t),Domain.inPExpCone())
        M.constraint('convex',Expr.sub(Expr.sub(emid,eleft),eright),Domain.lessThan(0.0))
        M.constraint('area',Expr.mul(0.5,Expr.dot(dy,Expr.add(x.slice(0,n-1),x.slice(1,n)))),
                     Domain.equalsTo(1.0))
        M.objective('obj',ObjectiveSense.Maximize,Expr.sum(t))

        M.solve()

        return x.level()
```

Output

n=1000

```
ITE PFEAS    DFEAS    GFEAS    PRSTATUS   POBJ              DOBJ              MU       TIME
0   1.2e+01  1.3e+00  8.3e+02  0.00e+00   -8.278383991e+02  0.000000000e+00   1.0e+00  0.02
1   8.8e+00  9.4e-01  6.7e+02  2.29e-01   -5.983537098e+02  6.025824922e+01   7.3e-01  0.03
....
85  1.5e-11  2.1e-11  2.2e-07  7.59e-01   -3.314420835e+03  -3.314420856e+03  1.6e-11  0.25
86  6.5e-12  4.5e-11  1.3e-07  8.34e-01   -3.314462710e+03  -3.314462720e+03  6.9e-12  0.25
87  2.4e-12  6.2e-11  7.9e-08  9.23e-01   -3.314482870e+03  -3.314482874e+03  2.6e-12  0.27
88  5.1e-13  1.3e-11  3.4e-08  9.69e-01   -3.314492646e+03  -3.314492646e+03  5.8e-13  0.27
```

X is a correlation matrix if

$$X \in \mathcal{C} := \{X \in K_s \mid \mathbf{diag} X = e\}.$$

Links:

- `https://en.wikipedia.org/wiki/Correlation_and_dependence`
- `https://nickhigham.wordpress.com/2013/02/13/the-nearest-correlation-matrix/`

Higham:

> *A correlation matrix is a symmetric matrix with unit diagonal and nonnegative eigenvalues. In 2000 I was approached by a London fund management company who wanted to find the nearest correlation matrix (NCM) in the Frobenius norm to an almost correlation matrix: a symmetric matrix having a significant number of (small) negative eigenvalues. This problem arises when the data from which the correlations are constructed is asynchronous or incomplete, or when models are stress-tested by artificially adjusting individual correlations. Solving the NCM problem (or obtaining a true correlation matrix some other way) is important in order to avoid subsequent calculations breaking down due to negative variances or volatilities, for example.*

The Frobenius norm

$$\|A\|_F := \sqrt{\sum_i \sum_j A_{ij}^2}.$$

Given a symmetric matrix A then the problem is

$$\begin{array}{ll} \text{minimize} & \|A - X\|_F \\ \text{subject to} & X \in \mathcal{K}_s. \end{array}$$

The problem

$$\begin{array}{lccl} \text{minimize} & t & & \\ \text{subject to} & [t; \text{vec}(A - X)] & \in & \mathcal{K}_q, \\ & \mathbf{diag} X & = & e, \\ & X & \succeq & 0, \end{array}$$

where

$$\text{vec}(U) = (U_{11}; \sqrt{2}U_{21}; \ldots, \sqrt{2}U_{n1}; U_{22}; \sqrt{2}U_{32}; \ldots, \sqrt{2}U_{n2}; \ldots; U_{nn})^T.$$

Python program

nearestcorr.py

```
import sys
import mosek
import mosek.fusion
from mosek.fusion import *
from mosek import LinAlg

"""
 Assuming that e is an NxN expression, return the lower triangular part as a vector.
"""
def vec(e):
    N = e.getShape().dim(0)

    msubi = range(N * (N + 1) // 2)
    msubj = [i * N + j for i in range(N) for j in range(i + 1)]
    mcof = [2.0**0.5 if i !=
            j else 1.0 for i in range(N) for j in range(i + 1)]

    S = Matrix.sparse(N * (N + 1) // 2, N * N, msubi, msubj, mcof)
    return Expr.mul(S, Expr.flatten(e))

def nearestcorr(A):
    N = A.numRows()

    # Create a model
    with Model("NearestCorrelation") as M:

        M.setLogHandler(sys.stdout)

        # Setting up the variables
        X = M.variable("X", Domain.inPSDCone(N))
        t = M.variable("t", 1, Domain.unbounded())

        # (t, vec (A-X)) \in Q
        v = vec(Expr.sub(A, X))
        M.constraint("C1", Expr.vstack(t, v), Domain.inQCone())

        # diag(X) = e
        M.constraint("C2", X.diag(), Domain.equalsTo(1.0))

        # Objective: Minimize t
        M.objective(ObjectiveSense.Minimize, t)
        M.solve()

        return X.level(), t.level()


if __name__ == '__main__':
    N = 5
    A = Matrix.dense(N, N, [0.0, 0.5, -0.1, -0.2, 0.5,
                            0.5, 1.25, -0.05, -0.1, 0.25,
                            -0.1, -0.05, 0.51, 0.02, -0.05,
                            -0.2, -0.1, 0.02, 0.54, -0.1,
                            0.5, 0.25, -0.05, -0.1, 1.25])

    gammas = [0.1 * i for i in range(11)]

    X, t = nearestcorr(A)
```

Output

```
ITE PFEAS    DFEAS    GFEAS    PRSTATUS   POBJ              DOBJ              MU       TIME
0   1.0e+00  1.0e+00  1.5e+00  0.00e+00   2.000000000e+00   0.000000000e+00   1.0e+00  0.00
1   3.6e-01  3.6e-01  9.2e-01  1.62e+00   1.600638694e+00   1.202603762e+00   3.6e-01  0.01
2   5.5e-02  5.5e-02  3.8e-01  1.40e+00   1.245660809e+00   1.204906169e+00   5.5e-02  0.01
3   5.8e-03  5.8e-03  1.5e-01  1.08e+00   1.258938758e+00   1.250139855e+00   5.8e-03  0.01
4   8.4e-05  8.4e-05  1.8e-02  1.01e+00   1.255732088e+00   1.255603125e+00   8.4e-05  0.01
5   1.7e-06  1.7e-06  2.5e-03  1.00e+00   1.255670538e+00   1.255667894e+00   1.7e-06  0.01
6   1.4e-07  1.4e-07  7.1e-04  1.00e+00   1.255667495e+00   1.255667284e+00   1.4e-07  0.01
7   5.1e-09  5.1e-09  1.4e-04  1.00e+00   1.255667167e+00   1.255667160e+00   5.1e-09  0.01
```

Fusion summary

- Implemented model is close to the paper version.
- Easy to add and remove constraints and variables.
- Excellent for rapid linear and conic model building.
- C++, Java, and .NET Fusion looks almost identical.
- Efficiency:
  - C++, Java, .NET: Usually low overhead.
  - Python: Models involving a lot of looping can be sluggish.

Fusion alternative: Matrix oriented API example

MATLAB toolbox example

MOSEK optimization toolbox for MATLAB includes

- A matrix orientated interface.
- Lower level than Fusion.
- linprog, quadprog, etc clones.

Serializing the model

First reformulation:

$$
\begin{array}{lll}
\text{maximize} & r^T x - \alpha s & \\
\text{subject to} & e^T x & = 1, \\
& Gx - t & = 0, \\
& [s; t] \in \mathcal{K}_q^n, & \\
& x \geq 0, &
\end{array}
$$

where

$$e = [1, \ldots, 1]^T.$$

A new variable:

$$\bar{x} = \begin{bmatrix} x \\ t \\ s \end{bmatrix} = [x; t; s]$$

Next reformulation

$$
\begin{array}{ll}
\text{maximize} & \begin{bmatrix} r^T & 0^T_{1\times n} & -\alpha \end{bmatrix} \bar{x} \\
\text{subject to} & \begin{bmatrix} e^T_{1\times n} & 0^T_{1\times n} & 0 \end{bmatrix} \bar{x} = 1 \\
& \begin{bmatrix} G & -I_{n\times n} & 0^T_{n\times 1} \end{bmatrix} \bar{x} = 0 \\
& \bar{x}_{2n+1} \geq \left\| \bar{x}_{(n+1):(2n)} \right\| \\
& \bar{x}_{1:n} \geq 0,
\end{array}
$$

MOSEK model

$$
\begin{array}{ll}
\text{maximize} & c^T x \\
\text{subject to} & l^c \leq Ax \leq u^c \\
& x \in K \\
& l^x \leq x \leq u^x
\end{array}
$$

Portfolio example in MATLAB:

```
[ret, res] = mosekopt('symbcon echo(0)');

r          = [ 0.1073,  0.0737,  0.0627 ]';
G          = [ [ 0.1667,  0.0232,  0.0013 ];...
               [ 0.0000,  0.1033, -0.0022 ];...
               [ 0.0000,  0.0000,  0.0338 ] ];
alphas     = [0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.75, 1.0, 1.5, 2.0, 3.0, 10.0];
n          = length(r);

clear prob;

% The the problem.
prob.a             = [[ones(1,n),zeros(1,n),0];...
                      [G,-speye(n),zeros(n,1)]];
prob.blc           = [1;zeros(n,1)];
prob.buc           = [1;zeros(n,1)];
prob.blx           = [zeros(n,1);-inf*ones(n+1,1)];

prob.cones.type    = [res.symbcon.MSK_CT_QUAD];
prob.cones.sub     = [(2*n+1),(n+1):(2*n)];
prob.cones.subptr  = [1];

% Compute the efficient frontier.
for i=1:length(alphas)
    alpha              = alphas(i);
    prob.c             = [r;zeros(n,1);-alpha];
    [ret,res]          = mosekopt('maximize echo(0)',prob);
    x                  = res.sol.itr.xx;
    fprintf('%.2e %.4e %.4e\n',alpha,r'*x(1:n),x(2*n+1));
end
```

MATLAB run

```
>> portfolio
alpha          ret            risk
0.00e+00       1.0730e-01     7.2173e-01
1.00e-01       1.0730e-01     1.6670e-01
2.00e-01       1.0730e-01     1.6670e-01
3.00e-01       8.0540e-02     6.8220e-02
4.00e-01       7.1951e-02     4.2329e-02
5.00e-01       6.9756e-02     3.7355e-02
7.50e-01       6.7660e-02     3.3827e-02
1.00e+00       6.6790e-02     3.2811e-02
1.50e+00       6.5984e-02     3.2139e-02
2.00e+00       6.5601e-02     3.1916e-02
3.00e+00       6.5221e-02     3.1758e-02
1.00e+01       6.4698e-02     3.1645e-02
```

MOSEK optimization toolbox for MATLAB summary

- Matrix orientated input.
- Models must be serialized.

Section 5

The algorithm for solving a conic optimization problem

Outline

An interior-point method for conic optimization:

- The simplified homogeneous model of Xu, Hung, and Ye.
  - Solves how to start the interior-point method.
  - Detecting infeasibility properly.
  - Improves numerical stability.
- The search direction.
  - NT direction.

The homogeneous model

Generalized Goldman-Tucker homogeneous model:

$$
\begin{aligned}
(H) \qquad Ax - b\tau &= 0, \\
A^T y + s - c\tau &= 0, \\
-c^T x + b^T y - \kappa &= 0, \\
(x;\tau) \in \bar{\mathcal{K}}, (s;\kappa) &\in \bar{\mathcal{K}}^*
\end{aligned}
$$

where

$$
\bar{\mathcal{K}} := \mathcal{K} \times \mathbb{R}_+ \quad \text{and} \quad \bar{\mathcal{K}}^* := \mathcal{K}^* \times \mathbb{R}_+.
$$

- $\mathcal{K}$ is Cartesian product of k convex cones.
- The homogeneous model always has a solution.
- Partial list of references:
  - Linear case: [5] (GT), [18] (YTM), [17] (XHY).
  - Nonlinear case: [11] (NTY).
  - Linear and semidefinite: Roos, Terlaky, Sturm, Zhang, and more.

Investigating the homogeneous model

Lemma

Let $(x^*, \tau^*, y^*, s^*, \kappa^*)$ be any feasible solution to (H), then

i)

$$(x^*)^T s^* + \tau^* \kappa^* = 0.$$

ii) If $\tau^* > 0$, then

$$(x^*, y^*, s^*)/\tau^*$$

is an optimal solution.

iii) If $\kappa^* > 0$, then at least one of the strict inequalities

$$b^T y^* > 0 \tag{1}$$

and

$$c^T x^* < 0 \tag{2}$$

holds. If the first inequality holds, then (P) is infeasible. If the second inequality holds, then (D) is infeasible.

Summary:

- Compute a nontrivial solution to (H).
- Provides required information.
- Illposed case:

$$\tau^* = \kappa^* = 0.$$

- Illposed case cannot occur for linear problems.
- Facial reduction information available in illposed case [12].

The algorithm

Let

$$x^{(0)}, \tau^{(0)}, y^{(0)}, s^{(0)}, \kappa^{(0)}$$

be a suitable starting point i.e. interior.

The search direction

$$\begin{aligned}
Ad_x - bd_\tau &= \eta(Ax^{(0)} - b\tau^{(0)}),\\
A^T d_y + d_s - cd_\tau &= \eta(A^T y^{(0)} + s^{(0)} - c\tau^{(0)}),\\
-c^T d_x + b^T d_y - d_\kappa &= \eta(-c^T x^{(0)} + b^T y^{(0)} - \kappa),\\
\bar{X}^{(0)}(W)^{-1} d_s + \bar{S}^{(0)} W d_x &= -\bar{X}^{(0)}\bar{S}^{(0)} e + \gamma\mu^{(0)} e,\\
\tau^{(0)} d_\kappa + \kappa^{(0)} d_\tau &= -\tau^{(0)}\kappa^{(0)} + \gamma\mu^{(0)}.
\end{aligned}$$

where

$$\bar{X}^{(0)} \text{ and } W$$

are symmetric positive definite matrices. Moreover,

$$\eta := \gamma - 1 \in [-1, 0].$$

New iterate:

$$\begin{bmatrix} x^{(1)} \\ \tau^{(1)} \\ y^{(1)} \\ s^{(1)} \\ \kappa^{(1)} \end{bmatrix} = \begin{bmatrix} x^{(0)} \\ \tau^{(0)} \\ y^{(0)} \\ s^{(0)} \\ \kappa^{(0)} \end{bmatrix} + \alpha \begin{bmatrix} d_x \\ d_\tau \\ d_y \\ d_s \\ d_\kappa \end{bmatrix}.$$

for some stepsize

$$\alpha \in (0, 1].$$

- New point must be interior!

Properties of the search direction

Lemma

$$
\begin{aligned}
Ax^{(1)} - b\tau^{(1)} &= (1+\alpha\eta)(Ax^{(0)} - b\tau^{(0)}),\\
A^T y^{(1)} + s^{(1)} - c\tau^{(1)} &= (1+\alpha\eta)(A^T y^{(0)} + s^{(0)} - c\tau^{(0)}),\\
-c^T x^{(1)} + b^T y^{(1)} - \kappa^{(1)} &= (1+\alpha\eta)(-c^T x^{(0)} + b^T y^{(0)} - \kappa^{(0)}),\\
d_x^T d_s^T + d_\tau d_\kappa &= 0,\\
(x^{(1)})^T s^{(1)} + \tau^{(1)}\kappa^{(1)} &= (1+\alpha\eta)((x^{(0)})^T s^{(0)} + \tau^{(0)}\kappa^{(0)}).
\end{aligned}
$$

Observations:

- The complementarity gap is reduced by a factor of $(1+\alpha\eta) \in [0,1]$.
- The infeasibility is reduced by the same factor.
- Highly advantageous property.
- Implies convergence.
- Polynomial complexity can be proven given some assumptions.

The scaling matrix

- The linear case:

$$W = \sqrt{\frac{s_j}{x_j}}.$$

- The symmetric cone case:
  - The Nesterov-Todd choice :

$$Wx = W^{-1}s$$

    leads to primal-dual symmetry!
- The nonsymmetric cone case:
  - Hard.
  - Tuncel [16], Myklebust and T. [9].
  - Skajaa and Ye [14], Serrano [13].
  - MOSEK: Primarily based on ideas of Tuncel.

Practical issues

- Step-size computation
  - Stepsize to boundary can be computed easily.
- Mehrotra predictor-corrector extension.
  - Estimate γ.
  - High-order correction.

Practical stopping criteria

- A solution

$$(x, y, s) = (x^{(k)}, y^{(k)}, s^{(k)})/\tau^{(k)}$$

is said to be primal-dual optimal solution if

$$\begin{aligned}
\left\| Ax^{(k)} - b\tau^{(k)} \right\|_\infty &\leq \varepsilon_p(1 + \|b\|_\infty)\tau^{(k)}, \\
\left\| A^T y^{(k)} + s^{(k)} - c\tau^{(k)} \right\|_\infty &\leq \varepsilon_d(1 + \|c\|_\infty)\tau^{(k)}, \\
\frac{|c^T x^{(k)} - b^T y^{(k)}|}{\tau^{(k)} + \max(|c^T x^{(k)}|, |b^T y^{(k)}|)} &\leq \varepsilon_g
\end{aligned}$$

where $\varepsilon_p, \varepsilon_d$ and ε_g all are small user specified constants.

- If
$$b^T y^{(k)} > 0 \text{ and } b^T y^{(k)} \varepsilon_i \geq \left\| A^T y^{(k)} + s^{(k)} \right\|_\infty$$
the problem is denoted to be primal infeasible and the certificate is $(y^{(k)}, s^{(k)})$ is reported.
- If
$$-c^T x^{(k)} > 0 \text{ and } -c^T x^{(k)} \varepsilon_i \geq \left\| A x^{(k)} \right\|_\infty$$
is said denoted to be dual infeasible and the certificate is $x^{(k)}$ is reported.

Observations about the stopping criterion

- Only an approximate solution is computed. (We work in finite precision anyway.)
- Stopping criterion is not God given but observed to work well in practice.
- Primal accuracy is proportional to $\|b\|_\infty$.
- Dual accuracy is proportional to $\|c\|_\infty$.
- Do and don'ts.
  - Scale the problem nicely.
  - Do not add large bounds.
  - Do not use large penalties in the objective.

Computation of the search direction

The computational most expensive operation in the algorithm is the search direction computation:

$$\begin{aligned} Ad_x - bd_\tau &= f^1, \\ A^T d_y + d_s - cd_\tau &= f^2, \\ -c^T d_x + b^T d_y - d_\kappa &= f^3, \\ \bar{X}^{(0)}(W)^{-1} d_s + \bar{S}^{(0)} W d_x &= f^4, \\ \tau^{(0)} d_\kappa + \kappa^{(0)} d_\tau &= f^5 \end{aligned}$$

where f^i represents an arbitrary right-hand side.
This implies

$$\begin{aligned} d_s &= (\bar{X}^{(0)}(W)^{-1})^{-1}(f^4 - \bar{S}^{(0)} W d_x) \\ &= (\bar{X}^{(0)}(W)^{-1})^{-1} f^4 - WW d_x, \\ d_\kappa &= (\tau^{(0)})^{-1}(f^5 - \kappa^{(0)} d_\tau). \end{aligned}$$

Hence,

$$\begin{aligned} Ad_x - bd_\tau &= f^1, \\ A^T d_y - WWd_x - cd_\tau &= \hat{f}^2, \\ -c^T d_x + b^T d_y + (\tau^{(0)})^{-1}\kappa^{(0)} d_\tau &= \hat{f}^3, \end{aligned}$$

and

$$d_x = -(WW)^{-1}(\hat{f}^2 - A^T d_y + cd_\tau).$$

Thus

$$\begin{aligned} A(WW)^{-1}A^T d_y - (b + A(WW)^{-1}c)d_\tau &= \hat{f}^1, \\ (b - A(WW)^{-1}c)^T d_y + (c^T(WW)^{-1}c + (\tau^{(0)})^{-1}\kappa^{(0)})d_\tau &= \tilde{f}^3. \end{aligned}$$

Given

$$M = A(WW)A^T = \sum_{k=1}^{r} A^k (W^k)^{-2} (A^k)^T,$$

and

$$\begin{aligned} Mv^1 &= (b + A(WW)^{-1}c), \\ Mv^2 &= \hat{f}^1 \end{aligned}$$

we reach the easy solvable linear system

$$\begin{aligned} d_y - v^1 d_\tau &= v^2, \\ (b - A(WW)^{-1}c)^T d_y + (c^T (WW)^{-1} c + (\tau^{(0)})^{-1} \kappa^{(0)}) d_\tau &= \tilde{f}^3. \end{aligned}$$

- The hard part is the linear equation systems involving M.
- $M = M^T$.
- M is positive definite.
- Use a sparse Cholesky factorization of M i.e.

$$M = LL^T.$$

Can we use sparse computations?

- But is M sparse? Yes, usually if
  - A^k is sparse.
  - A^k contains no dense columns.
  - No high dimensional cones.
- M is usually very sparse in the linear and conic quadratic cases.

Observations

- Fewer rows in A tends to be better.
  - Does the primal or the dual has fewest rows.
- Big cones and/or dense columns in A are trouble some.
  - Dense rows are not problematic.
  - It is possible to deal with dense columns and large cones
    - See for instance [1] for details.

Forming and factorizing M

- Formed using multiple threads (a sparse syrk).
- Compute a Cholesky.
- Employ approximate minimum degree and/or GP ordering to obtain a good ordering.
- Implemented a high-performance sparse Cholesky.
  - Exploit INTEL MKL BLAS for dense matrix multiplications.
  - Parallelized (using Cilk Plus).
- **VERY EFFICENT**.
  - LIttle room for iterative methods e.g. CG.

Other practical issues

- Employs presolve to reduce problem size.
- Problem is dualized when considered worthwhile. (why?).
- Exploit problem structure:
  - Upper bounds on linear variables: $x_j \leq u_j$.
  - Fixed variables: $x_j = u_j$.

Section 6

Benchmark results

Conic quadratic optimization

- Hardware: Intel based server (Xeon Gold 6126 2.6 GHz, 12 core).
- MOSEK: 8.1.0.34.
- Threads: 8 threads is used in test to simulate a typical user environment.
- All timing results t are in wall clock seconds.
- Test problems: Public (e.g `cblib.zib.de`) and customer supplied.

Conic quadratic optimization

Optimized problems

Name	# con.	# cone	# var.	# mat. var.
2D_43_dual	10645	10081	50965	0
2D_43_primal	10645	10081	50965	0
Barcelona_p4	83417	5042	245048	0
a_case118	612	188	3082	0
beam30	113627	30	115817	0
c-nql180	53128	32400	117450	0
c-qssp180	64799	65338	260278	0
c-traffic-36	2740	1331	5401	0
chainsing-1000000	2999994	2999994	9999982	0
frozenpizza_2_cap100_conic_mc	894	670	5573	0
igl-eth-schuller-optProblemSmall	20000	1	76560	0
igl_n	239	122	402	0
multibody-3	12000	1	215012	0
paper-large-scale-co-for-wifi-net-50	10000	101	25151	0
pp-n100000-d10000	1	100000	300001	0
primal_minball_100000	3	100000	300000	0
soccer-trajectory-20160118-ticket11770	55322	82980	359576	0
sssd-strong-30-8	110	24	368	0
swiss_quant_challenge_100_100000_0_primal_quad	100	1	100102	0
tv_employee_200x200_sigma10	79600	40000	159600	0
yuriv2cm	951	421	4832	0
yuriy4	908	421	3496	0
2013_firL2a	10000	1	10002	0
2013_firLinf	2001	19952	59855	0
2013_wbNRL	1041	9	40450	0
igl-eth-schuller-optProblem2	297493	1	1080542	0

Conic quadratic optimization

Result

Name	P. obj.	# sig. fig.	# iter	time(s)
2D_43_dual	-8.7829000557e-01	13	22	1.0
2D_43_primal	8.7829000557e-01	13	22	1.0
Barcelona_p4	1.4838295332e+00	8	28	23.8
a_case118	1.2542408065e+02	10	24	0.1
beam30	1.0196786009e+01	8	51	194.1
c-nql180	-9.2764832251e-01	8	11	4.8
c-qssp180	-6.6384468642e+00	9	13	5.0
c-traffic-36	-5.3902456727e+03	10	22	0.3
chainsing-1000000	6.0504870160e+05	8	15	88.4
frozenpizza_2_cap100_conic_mc	1.0381566979e+00	11	24	0.3
igl-eth-schuller-optProblemSmall	7.8481282458e+01	9	30	4.3
igl_n	-7.8587340784e+03	8	19	0.1
multibody-3	-2.9552479021e-01	7	17	5.7
paper-large-scale-co-for-wifi-net-50	-2.7585519719e+00	7	11	10.9
pp-n100000-d10000	2.1957043097e+07	8	18	2.7
primal_minball_100000	4.7675622219e+00	11	21	2.2
soccer-trajectory-20160118-ticket11770	3.6231654545e+01	8	25	5.7
sssd-strong-30-8	3.5848195609e+05	9	19	0.0
swiss_quant_challenge_100_100000_0_primal_quad	4.2175339166e+03	9	10	4.6
tv_employee_200x200_sigma10	1.3368471531e+05	9	21	4.3
yuriv2cm	4.0126130625e-03	5	24	4.2
yuriy4	5.1508485788e-03	5	23	3.6
2013_firL2a	-1.4367664349e-01	8	3	29.3
2013_firLinf	-1.0022267370e-02	13	17	195.4
2013_wbNRL	-3.9953991287e-05	8	11	12.4
igl-eth-schuller-optProblem2	2.2982849956e+01	7	20	59.9

A huge conic quadratic optimization

- Hardware: Intel based server (2*E5-2687W, 3Ghz, 12 core).
- MOSEK: 8.1.0.34.
- Threads: 24
- All timing results t are in wall clock seconds.

A huge conic quadratic optimization

Optimized problems

Name	# con.	# cone	# var.	# mat. var.
huge	15822	52141	327653	0

- Number of nonzero: 2208749451 (2.2 billions)
- Flops per iteration: 3.6e13

Conic quadratic optimization

Result

Name	P. obj.	# sig. fig.	# iter	time(s)
huge	-1.2480032495e+05	9	8	1587.2

Exponential optimization

- Hardware: Intel based server. (Xeon Gold 6126 2.6 GHz, 12 core)
- MOSEK: Version 9 (alpha). Highly experimental!
- Threads: 8 threads is used in test to simulate a typical user environment.
- All timing results t are in wall clock seconds.
- Test problems: Public (e.g `cblib.zib.de`) and customer supplied.

Exponential optimization

Optimized problems

Name	# con.	# cone	# var.	# mat. var.
task_dopt3	1600	26	376	2
task_dopt16	1600	26	376	2
entolib_a_bd	26	4695	14085	0
entolib_ento2	26	4695	14085	0
task_dopt10	1600	26	376	2
task_dopt17	1600	26	376	2
entolib_ento50	28	5172	15516	0
entolib_a_36	37	7497	22491	0
entolib_ento3	28	5172	15516	0
task_dopt12	1600	26	376	2
entolib_ento48	31	15364	46092	0
task_dopt21	1600	26	376	2
entolib_a_25	37	6196	18588	0
entolib_ento26	28	7915	23745	0
entolib_ento45	37	9108	27324	0
entolib_a_26	37	9035	27105	0
entolib_ento25	28	10142	30426	0
entolib_a_16	37	8528	25584	0
entolib_a_56	37	9702	29106	0
exp-ml-scaled-20000	19999	20000	79998	0
entolib_entodif	40	12691	38073	0
exp-ml-20000	19999	20000	79998	0

Exponential cone optimization

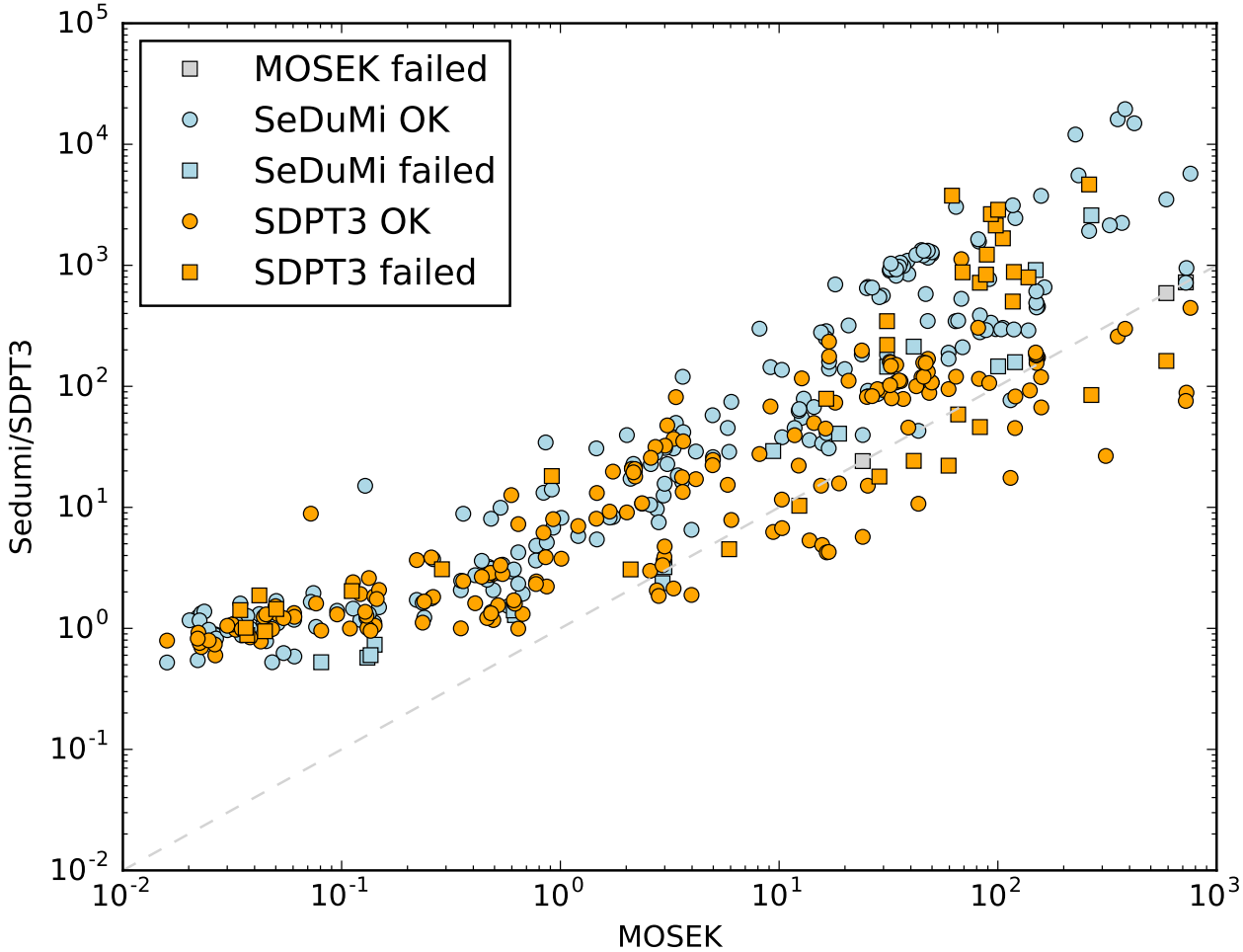
Result

Name	P. obj.	# sig. fig.	# iter	time(s)
task_dopt3	1.5283637408e+01	9	19	0.8
task_dopt16	1.3214504234e+01	9	19	0.8
entolib_a_bd	-1.1354775044e+01	7	87	0.9
entolib_ento2	-1.1354775044e+01	7	87	0.9
task_dopt10	1.4373687081e+01	9	23	1.0
task_dopt17	1.6884207279e+01	9	26	1.1
entolib_ento50	-6.5012062421e+00	7	118	1.3
entolib_a_36	2.4244266552e+00	4	79	1.3
entolib_ento3	-6.5012062421e+00	7	118	1.3
task_dopt12	2.3128677474e+01	10	40	1.6
entolib_ento48	-1.0455733646e+01	7	55	1.7
task_dopt21	2.5769926047e+01	9	47	1.9
entolib_a_25	-7.9656915830e+00	7	152	2.1
entolib_ento26	-1.1576975407e+01	7	129	2.1
entolib_ento45	-8.7854360451e+00	7	122	2.5
entolib_a_26	-7.6584841226e+00	8	126	2.6
entolib_ento25	-7.2807252375e+00	7	135	2.8
entolib_a_16	-4.7658117480e+00	6	162	3.0
entolib_a_56	-8.2835713863e+00	6	140	3.2
exp-ml-scaled-20000	-3.3120019648e+00	10	63	3.1
entolib_entodif	-6.3525985469e+00	7	175	4.8
exp-ml-20000	-1.9793436580e+04	9	144	7.6

- We use a subset of Mittelmann's benchmark-set + customer provided problems.
- MOSEK on 4 threads, MATLAB up to 20 threads.
- Problems are categorized as **failed** if
  - MOSEK returns unknown solution status.
  - SeDuMi returns `info.numerr==2`.
  - SDPT3 returns `info.termcode` $\notin [0, 1, 2]$ and `norm(info.dimacs,Inf)>1e-5`.

Semidefinite problems

Comparison with SeDuMi and SDPT3



Solution time for MOSEK v8.0.0.42 vs SeDuMi/SDPT3 on 234 problems.

Semidefinite problems

Comparison with SeDuMi and SDPT3

	small			medium		
	MOSEK	SeDuMi	SDPT3	MOSEK	SeDuMi	SDPT3
Num.	127	127	127	63	63	63
Firsts	116	1	10	47	0	16
Total time	218.2	1299.5	843.6	2396.0	32709.8	5679.5

	large			fails		
	MOSEK	SeDuMi	SDPT3	MOSEK	SeDuMi	SDPT3
Num.	44	44	44	6	27	47
Firsts	31	0	13			
Total time	9083.8	119818.1	35268.2			

- MOSEK is fastest on average with fewest failures.
- SeDuMi is almost always slower than MOSEK.

Section 7

Summary

The take home message

- Extremely disciplined modelling:
  - Aka. conic modelling.
  - Can be used to express most convex optimization models.
  - Has many advantages e.g. simple, explicit and structurally convex.
- MOSEK:
  - Solves (mixed integer) conic optimization problems.

Further information

- Documenation at
  `https://www.mosek.com/documentation/`
  - Manuals for interfaces.
  - Modelling cook book.
  - White papers.
- Examples
  - Tutorials at Github:
    `https://github.com/MOSEK/Tutorials`

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