On Facial Reduction in the MOSEK Conic Optimizer

ICCOPT 2016, Tokyo, Japan

Henrik A. Friberg

www.mosek.com

The self-dual embedding

P$(\mathcal{K})$:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \in \mathcal{K}, \end{array}$$

D$(\mathcal{K})$:

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & c - A^T y = s, \\ & (s, y) \in \mathcal{K}^* \times \mathbb{R}^m, \end{array}$$

for a non-empty, closed, convex cone $\mathcal{K}$.

The self-dual embedding

P($\mathcal{K}$):

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \in \mathcal{K}, \end{array}$$

D($\mathcal{K}$):

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & c - A^T y = s, \\ & (s, y) \in \mathcal{K}^* \times \mathbb{R}^m, \end{array}$$

H($\mathcal{K}$):

$$\begin{array}{rcl} Ax - b\tau & = & 0, \\ -A^T y - s + c\tau & = & 0, \\ b^T y - c^T x - \kappa & = & 0, \\ \multicolumn{3}{c}{(x, s, y, \tau, \kappa) \in \mathcal{K} \times \mathcal{K}^* \times \mathbb{R}^m \times \mathbb{R}_+ \times \mathbb{R}_+.} \end{array}$$

Observations

- Complementarity in **H**($\mathcal{K}$) by definition: $x^T s = \tau\kappa = 0$.
- $\tau > 0 \Rightarrow$ Complementary solution for **P**($\mathcal{K}$) and **D**($\mathcal{K}$).
- $\kappa > 0 \Rightarrow$ Improving ray for **P**($\mathcal{K}$) or **D**($\mathcal{K}$).

The self-dual embedding

Third case: $\tau = \kappa = 0$

In Permenter, Friberg, and Andersen, *Solving conic optimization problems via self-dual embedding and facial reduction: a unified approach*, 2015:

$$\begin{aligned} Ax &= 0, \\ A^T y + s &= 0, \\ b^T y - c^T x &= 0, \\ (x, s, y, \tau, \kappa) \in \mathcal{K} \times \mathcal{K}^* \times \mathbb{R}^m \times 0 \times 0. \end{aligned}$$

The self-dual embedding

Third case: $\tau = \kappa = 0$

In Permenter, Friberg, and Andersen, *Solving conic optimization problems via self-dual embedding and facial reduction: a unified approach*, 2015:

$$
\begin{aligned}
Ax &= 0, \\
A^T y + s &= 0, \\
b^T y - c^T x &= 0, \\
(x, s, y, \tau, \kappa) &\in \mathcal{K} \times \mathcal{K}^* \times \mathbb{R}^m \times 0 \times 0.
\end{aligned}
$$

Points like $(x, s, y, \tau, \kappa) = (0, 0, 0, 0, 0)$ will never be meaningful!

The self-dual embedding

Third case: $\tau = \kappa = 0$

In Permenter, Friberg, and Andersen, *Solving conic optimization problems via self-dual embedding and facial reduction: a unified approach*, 2015:

$$\begin{aligned} Ax &= 0, \\ A^T y + s &= 0, \\ b^T y - c^T x &= 0, \end{aligned}$$
$$(x, s, y, \tau, \kappa) \in \mathcal{K} \times \mathcal{K}^* \times \mathbb{R}^m \times 0 \times 0.$$

Key note from Theorem 1

Considering relint $\mathbf{H}(\mathcal{K})$, then $\tau = \kappa = 0$ is exceptional.

- If $\mathbf{P}(\mathcal{K})$ is troublesome: $\mathcal{K} \to \mathcal{K} \cap s^\perp$ (facial reduction).
- If $\mathbf{D}(\mathcal{K})$ is troublesome: $\mathcal{K}^* \to \mathcal{K}^* \cap x^\perp$ (facial reduction).

The self-dual embedding

The self-dual embedding method
can solve all instances!

Assuming oracle for relint $\mathbf{H}(\mathcal{K})$ points.

An oracle for relint $\mathbf{H}(\mathcal{K})$ points?

In theory:

Yes, if tracking the central path with infinite precision.

In theory:

Yes, if tracking the central path with infinite precision.

In practice:

Issues in exceptional $\tau = \kappa = 0$ case:

- Lack of strict complementarity
  $\Rightarrow$ **slower and less accurate convergence.**
- Ill-posedness (i.e., instability to infinitesimal perturbations)
  $\Rightarrow$ **overlapping termination criteria.**
- Interior-point method
  $\Rightarrow$ **when are τ and κ small enough?**
  $\Rightarrow$ **have to be careful about orthogonal complements, e.g., when computing $\mathcal{K} \cap s^{\perp}$.**

Facial reduction certificate of $\mathbf{D}(\mathcal{K})$

In theory:

$$c^T x = 0, \quad Ax = 0, \quad x \in \mathcal{K} \setminus (\mathcal{K}^*)^\perp.$$

Facial reduction certificate of $\mathbf{D}(\mathcal{K})$

In theory:

$$c^T x = 0, \quad Ax = 0, \quad x \in \mathcal{K} \setminus (\mathcal{K}^*)^\perp.$$

Facial reduction certificate of $\mathbf{D}(\mathcal{K})$

In theory:

$$c^T x = 0, \quad Ax = 0, \quad x \in \mathcal{K} \setminus (\mathcal{K}^*)^\perp.$$

Facial reduction certificate of $\mathbf{D}(\mathcal{K})$

In theory:

$$c^T x = 0, \quad Ax = 0, \quad x \in \mathcal{K} \setminus (\mathcal{K}^*)^\perp.$$

Facial reduction certificate of $\mathbf{D}(\mathcal{K})$

In theory:

$$c^T x = 0, \quad Ax = 0, \quad x \in \mathcal{K} \setminus (\mathcal{K}^*)^\perp.$$

This implies $x^T(c - A^T y) = x^T s \iff x^T s = 0$ *and thus*

$$s \in \mathcal{K}^* \quad \longrightarrow \quad s \in \mathcal{K}^* \cap x^\perp$$

Facial reduction certificate of $\mathbf{D}(\mathcal{K})$

In theory:

$$c^T x = 0, \quad Ax = 0, \quad x \in \mathcal{K} \setminus (\mathcal{K}^*)^\perp.$$

This implies $x^T(c - A^T y) = x^T s \iff x^T s = 0$ *and thus*

$$s \in \mathcal{K}^* \quad \longrightarrow \quad s \in \mathcal{K}^* \cap x^\perp \text{ (a proper face of } \mathcal{K}^*\text{)}$$

Facial reduction certificate of $\mathbf{D}(\mathcal{K})$

In theory:

$$c^T x = 0, \quad Ax = 0, \quad x \in \mathcal{K} \setminus (\mathcal{K}^*)^\perp.$$

This implies $x^T(c - A^T y) = x^T s \iff x^T s = 0$ *and thus*

$$s \in \mathcal{K}^* \quad \longrightarrow \quad s \in \mathcal{K}^* \cap x^\perp \text{ (a proper face of } \mathcal{K}^*\text{)}$$

In practice:

- Have $x \in \mathcal{K}$.
- Need negligible $|c^T x|$ and $\|Ax\|$.
- Need significant $\operatorname{dist}(x, (\mathcal{K}^*)^\perp)$. $(= \|x\|$ for proper cones$)$

Facial reduction certificate of $\mathbf{P}(\mathcal{K})$

In theory:

$$b^T y = 0, \quad A^T y + s = 0, \quad s \in \mathcal{K}^* \setminus \mathcal{K}^\perp$$

This implies $y^T Ax = y^T b \iff s^T x = 0$ *and thus*

$$x \in \mathcal{K} \longrightarrow x \in \mathcal{K} \cap s^\perp \text{ (a proper face of } \mathcal{K})$$

In practice:

- Have $s \in \mathcal{K}^*$.
- Need negligible $|b^T y|$ and $\|A^T y + s\|$.
- Need significant $\operatorname{dist}(s, (\mathcal{K}^*)^\perp)$. $(= \|s\|$ for proper cones$)$

Facial reduction certificate of $\mathbf{P}(\mathcal{K})$

In theory:

$$b^T y = 0, \quad A^T y + s = 0, \quad s \in \mathcal{K}^* \setminus \mathcal{K}^\perp$$

This implies $y^T A x = y^T b \iff s^T x = 0$ *and thus*

$$x \in \mathcal{K} \longrightarrow x \in \mathcal{K} \cap s^\perp \text{ (a proper face of } \mathcal{K}\text{)}$$

In practice:

- Have $s \in \mathcal{K}^*$.
- Need negligible $|b^T y|$ and $\|A^T y + s\|$.
- Need significant $\mathrm{dist}(s, (\mathcal{K}^*)^\perp)$. ($\approx \|y\|$ significant)

Matrix A needs to be well-scaled and of full row rank.

Matrix A needs to be well-scaled and of full row rank.

Matrix A needs to be well-scaled and of full row rank.

If not satisfied, we may get $\|A^T y + s\| = 0$ for significant $\|y\|$ and negligible $\|s\|$.

Matrix A needs to be well-scaled and of full row rank.

If not satisfied, we may get $\|A^T y + s\| = 0$ for significant $\|y\|$ and negligible $\|s\|$. But then:

Linear row dependency certificate of $\mathbf{P}(\mathcal{K})$

$$y^T A = 0, \quad y^T b = 0, \quad y \neq 0$$

Our proposal

In principle:

1. Negligible: $\|Ax\|$ **and** $|c^T x|$.
2. Negligible: $\|A^T y + s\|$ **and** $|b^T y|$.
3. Significant: $\|x\|$ **or** $\|y\|$.

In implementation:

1. $\max\left(\|Ax\|, |c^T x|\right) \leq \epsilon \max(1, \|x\|)$.
2. $\max\left(\|A^T y + s\|, |b^T y|\right) \leq \epsilon \max(1, \|y\|)$.
3. $\max\left(\|Ax\|, |c^T x|\right) < \epsilon \|x\|$ **or**
   $\max\left(\|A^T y + s\|, |b^T y|\right) < \epsilon \|y\|$.

H. Waki, M. Nakata and M. Muramatsu, "Strange Behaviors of Interior-point Methods for Solving Semidefinite Programming Problems in Polynomial Optimization", 2012:

`unboundDim1R*`

- Dual is SDP-relaxation of POP (minimized).
- Primal ill-posed with (p;d) $= (0.0; 0.0^{+})$.

H. Waki, M. Nakata and M. Muramatsu, "Strange Behaviors of Interior-point Methods for Solving Semidefinite Programming Problems in Polynomial Optimization", 2012:

`unboundDim1R*`

- Dual is SDP-relaxation of POP (minimized).
- Primal ill-posed with (p;d) $=$ (0.0; 0.0^{+}).
- Higher-order Lasserre hierarchy means smaller perturbation exist after which (p;d) $=$ (1.0; 1.0).

H. Waki, M. Nakata and M. Muramatsu, "Strange Behaviors of Interior-point Methods for Solving Semidefinite Programming Problems in Polynomial Optimization", 2012:

unboundDim1R*

- Dual is SDP-relaxation of POP (minimized).
- Primal ill-posed with (p;d) = (0.0; 0.0^{+}).
- Higher-order Lasserre hierarchy means smaller perturbation exist after which (p;d) = (1.0; 1.0).

*	**Status**
1	Complementary solution of value 0.0
2	Near to complementary solution of value 0.0
3-4	Ill-posed certificate
5-10	Complementary solution of value 1.0

The MOSEK Conic Optimizer

unboundDim1R 1

```
ITE PFEAS    DFEAS    GFEAS    MU
0   1.0e+00  1.0e+00  2.0e+00  1.0e+00
1   3.1e-02  3.1e-02  1.1e+00  3.1e-02
2   4.8e-05  4.8e-05  2.4e-02  4.8e-05
3   1.1e-09  1.1e-09  2.2e-09  1.1e-09
Interior-point optimizer terminated. Time: 0.00.

Problem status  : PRIMAL_AND_DUAL_FEASIBLE
Solution status : OPTIMAL
Primal.  obj: -1.3336443272e-09   nrm: 1e+00   con: 1e-09      var: 0e+00
                                                            barvar: 2e-20

Dual.    obj: 7.0491750159e-10    nrm: 2e+00   con: 0e+00      var: 1e-09
                                                            barvar: 1e-09
```

The MOSEK Conic Optimizer

unboundDim1R 2

```
ITE PFEAS    DFEAS    GFEAS    MU
0   2.0e+00  2.0e+00  1.0e+00  1.0e+00
1   2.9e-01  2.9e-01  1.7e-01  1.5e-01
2   5.2e-02  5.2e-02  7.2e-02  2.6e-02
...
37  4.1e-21  5.4e-20  4.7e-19  2.0e-21
38  4.1e-21  5.4e-20  4.7e-19  2.0e-21
39  4.1e-21  5.4e-20  4.7e-19  2.0e-21
Interior-point optimizer terminated. Time: 0.01.

Problem status  : PRIMAL_AND_DUAL_FEASIBLE
Solution status : NEAR_OPTIMAL
Primal.  obj: 2.1214425554e-04   nrm: 1e+00   con: 3e-13   barvar: 0e+00
Dual.    obj: 1.9285790417e-04   nrm: 3e+08   con: 0e+00   barvar: 3e-01
```

The MOSEK Conic Optimizer

unboundDim1R 2 (note: overlapping termination criteria)

```
ITE PFEAS    DFEAS    GFEAS    MU
0   2.0e+00  2.0e+00  1.0e+00  1.0e+00
1   2.9e-01  2.9e-01  1.7e-01  1.5e-01
2   5.2e-02  5.2e-02  7.2e-02  2.6e-02
...
37  4.1e-21  5.4e-20  4.7e-19  2.0e-21
38  4.1e-21  5.4e-20  4.7e-19  2.0e-21
39  4.1e-21  5.4e-20  4.7e-19  2.0e-21
Interior-point optimizer terminated. Time: 0.01.

Problem status  : PRIMAL_AND_DUAL_FEASIBLE
Solution status : NEAR_OPTIMAL
Primal.  obj: 2.1214425554e-04    nrm: 1e+00    con: 3e-13    barvar: 0e+00
Dual.    obj: 1.9285790417e-04    nrm: 3e+08    con: 0e+00    barvar: 3e-01
```

unboundDim1R 3

```
ITE PFEAS    DFEAS    GFEAS    MU
0   2.0e+00  2.0e+00  1.0e+00  1.0e+00
1   3.3e-01  3.3e-01  1.5e-01  1.7e-01
2   5.5e-02  5.5e-02  6.7e-02  2.8e-02
...
30  7.9e-17  2.7e-15  6.6e-16  4.0e-17
31  7.9e-17  2.7e-15  6.6e-16  4.0e-17
32  7.9e-17  2.7e-15  6.6e-16  4.0e-17
Interior-point optimizer terminated. Time: 0.01.

Interior-point solution summary
Problem status  : ILL_POSED
Solution status : PRIMAL_ILLPOSED_CER
Dual.    obj: 1.3716138286e-08    nrm: 5e+00    con: 0e+00   barvar: 1e-07
```

H. Waki: "How to generate weakly infeasible semidefinite programs via Lasserre's relaxations for polynomial optimization", 2012:

`CompactDim2R*`

- Dual is SDP-relaxation of POP (minimized).
- Primal ill-posed with (p;d) $= (-\infty; -\infty^{+})$.

H. Waki: "How to generate weakly infeasible semidefinite programs via Lasserre's relaxations for polynomial optimization", 2012:

`CompactDim2R*`

- Dual is SDP-relaxation of POP (minimized).
- Primal ill-posed with (p;d) $= (-\infty; -\infty^{+})$.
- Higher-order Lasserre hierarchy means smaller perturbation exist after which (p;d) = (-1.5; -1.5).

*	**Status**
1	Primal infeasibility certificate
2	Ill-posed certificate
3-10	Complementary solution of value -1.5

```
ITE PFEAS    DFEAS    GFEAS    MU
0   2.0e+00  1.0e+00  2.0e+00  1.0e+00
1   4.2e-01  2.1e-01  4.2e-01  2.1e-01
2   7.1e-02  3.6e-02  5.3e-02  3.6e-02
...
23  1.2e-13  8.0e-14  4.9e-15  6.2e-14
24  2.6e-14  1.6e-14  3.1e-16  1.3e-14
25  3.5e-15  3.1e-15  5.3e-17  1.8e-15
Interior-point optimizer terminated. Time: 0.00.

Problem status  : PRIMAL_INFEASIBLE
Solution status : PRIMAL_INFEASIBLE_CER
Dual.    obj: -4.2224567689e-05   nrm: 3e+00   con: 0e+00       var: 2e-10
                                                              barvar: 3e-10
```

The MOSEK Conic Optimizer

CompactDim2R 1 (note: overlapping termination criteria)

```
ITE PFEAS    DFEAS    GFEAS    MU
0   2.0e+00  1.0e+00  2.0e+00  1.0e+00
1   4.2e-01  2.1e-01  4.2e-01  2.1e-01
2   7.1e-02  3.6e-02  5.3e-02  3.6e-02
...
23  1.2e-13  8.0e-14  4.9e-15  6.2e-14
24  2.6e-14  1.6e-14  3.1e-16  1.3e-14
25  3.5e-15  3.1e-15  5.3e-17  1.8e-15
Interior-point optimizer terminated. Time: 0.00.

Problem status  : PRIMAL_INFEASIBLE
Solution status : PRIMAL_INFEASIBLE_CER
Dual.    obj: -4.2224567689e-05   nrm: 3e+00   con: 0e+00      var: 2e-10
                                                            barvar: 3e-10
```

```
ITE PFEAS    DFEAS    GFEAS    MU
0   2.0e+00  1.5e+00  1.5e+00  1.0e+00
1   4.4e-01  3.3e-01  3.8e-01  2.2e-01
2   9.2e-02  6.9e-02  2.3e-01  4.6e-02
...
25  1.8e-15  3.6e-15  5.0e-16  9.0e-16
26  1.8e-15  3.6e-15  5.0e-16  9.0e-16
27  1.8e-15  3.6e-15  5.0e-16  9.0e-16
Interior-point optimizer terminated. Time: 0.01.

Problem status  : ILL_POSED
Solution status : PRIMAL_ILLPOSED_CER
Dual.    obj: -1.7098776698e-06   nrm: 7e+00   con: 0e+00   barvar: 1e-08
```

$$
\begin{array}{lll}
\text{maximize} & \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \overline{X} \right\rangle - 0.5x_0 + 0.5x_1 + 0.5x_2 & \\
\text{subject to} & \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \overline{X} \right\rangle - x_1 & = 1.0 \\
& \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \overline{X} \right\rangle - x_2 & = 1.0 \\
& \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \overline{X} \right\rangle & = 0.0 \\
& \left\langle \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \overline{X} \right\rangle + x_0 & = 0.0 \\
& \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \overline{X} \right\rangle & = 0.0 \\
& \overline{X} \in \mathcal{S}_+^3 & x \in \mathbb{R}_+^3
\end{array}
$$

CompactDim2R 1 : Facial reduction in action

$$
\begin{array}{lll}
\text{maximize} & \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \overline{X} \right\rangle - 0.5x_0 + 0.5x_1 + 0.5x_2 & \\
\text{subject to} & \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \overline{X} \right\rangle - x_1 = 1.0 & [\,10^{-5}\,] \\
& \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \overline{X} \right\rangle - x_2 = 1.0 & [\,10^{-5}\,] \\
& \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \overline{X} \right\rangle = 0.0 & [\,3\,] \\
& \left\langle \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \overline{X} \right\rangle + x_0 = 0.0 & [\,-0.4\,] \\
& \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \overline{X} \right\rangle = 0.0 & [\,3\,] \\
& \overline{X} \in \mathcal{S}^3_+ \begin{bmatrix} -3 & & \\ 0.4 & -3 & \\ 10^{-5} & 10^{-5} & 10^{-10} \end{bmatrix} \quad x \in \mathbb{R}^3_+ \begin{bmatrix} 0.4 \\ 10^{-5} \\ 10^{-5} \end{bmatrix} &
\end{array}
$$

CompactDim2R 1: Facial reduction in action

$$\text{maximize} \quad \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \overline{X} \right\rangle - 0.5x_0 + 0.5x_1 + 0.5x_2$$

$$\text{subject to} \quad \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \overline{X} \right\rangle - x_1 = 1.0 \quad [\,10^{-5}\,]$$

$$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \overline{X} \right\rangle - x_2 = 1.0 \quad [\,10^{-5}\,]$$

$$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \overline{X} \right\rangle = 0.0 \quad [\,3\,]$$

$$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \overline{X} \right\rangle + x_0 = 0.0 \quad [\,-0.4\,]$$

$$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \overline{X} \right\rangle = 0.0 \quad [\,3\,]$$

$$\overline{X} \in \mathcal{S}^3_+ \quad \begin{bmatrix} -3 & & \\ 0.4 & -3 & \\ 10^{-5} & 10^{-5} & 10^{-10} \end{bmatrix} \qquad x \in \mathbb{R}^3_+ \quad \begin{bmatrix} 0.4 \\ 10^{-5} \\ 10^{-5} \end{bmatrix}$$

CompactDim2R 1: Facial reduction in action

$$
\begin{array}{lrcll}
\text{maximize} & \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \overline{X} \right\rangle - 0.5x_0 + 0.5x_1 + 0.5x_2 & & & \\
\text{subject to} & \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \overline{X} \right\rangle - x_1 & = & 1.0 & [\, 10^{-5} \,] \\
 & \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \overline{X} \right\rangle - x_2 & = & 1.0 & [\, 10^{-5} \,] \\
 & \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \overline{X} \right\rangle & = & 0.0 & [\, 3 \,] \\
 & \left\langle \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \overline{X} \right\rangle + x_0 & = & 0.0 & [\, -0.4 \,] \\
 & \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \overline{X} \right\rangle & = & 0.0 & [\, 3 \,] \\
\end{array}
$$

$$
\overline{X} \in \mathcal{S}^3_+ \quad \begin{bmatrix} -3 & & \\ 0.4 & -3 & \\ 10^{-5} & 10^{-5} & 10^{-10} \end{bmatrix} \qquad x \in \mathbb{R}^3_+ \quad \begin{bmatrix} 0.4 \\ 10^{-5} \\ 10^{-5} \end{bmatrix}
$$

Liu and Pataki, *"Exact duals and short certificates of infeasibility and weak infeasibility in conic linear programming"*, 2015:

n=10, m=10	Strongly/weakly infeasible mix		Weakly infeasible	
	Clean	Messy	Clean	Messy
SEDUMI	87	27	0	0
SDPT3	10	5	0	0
MOSEK 7-series	63	17	0	0
MOSEK 8.0.0.33(BETA)	**98**w89	**65**w64	**100**w100	0
PP+SEDUMI	100	27	100	0

Liu and Pataki, *"Exact duals and short certificates of infeasibility and weak infeasibility in conic linear programming"*, 2015:

n=10, m=20	Strongly/weakly infeasible mix		Weakly infeasible	
	Clean	Messy	Clean	Messy
SEDUMI	100	100	1	0
SDPT3	100	96	0	0
MOSEK 7-series	100	100	11	0
MOSEK 8.0.0.33(BETA)	**100**w1	**69**w8	**98**w98	0
PP+SEDUMI	0	100	0	0