



On Facial Reduction in the MOSEK Conic Optimizer

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P(\mathcal{K}):

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \in \mathcal{K}, \end{array}$$

D(\mathcal{K}):

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & c - A^T y = s, \\ & (s, y) \in \mathcal{K}^* \times \mathbb{R}^m, \end{array}$$

for a non-empty, closed, convex cone \mathcal{K} .





P(\mathcal{K}):

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && Ax = b, \\ & && x \in \mathcal{K}, \end{aligned}$$

D(\mathcal{K}):

$$\begin{aligned} & \text{maximize} && b^T y \\ & \text{subject to} && c - A^T y = s, \\ & && (s, y) \in \mathcal{K}^* \times \mathbb{R}^m, \end{aligned}$$

H(\mathcal{K}):

$$\begin{aligned} Ax - b\tau &= 0, \\ -A^T y - s + c\tau &= 0, \\ b^T y - c^T x - \kappa &= 0, \\ (x, s, y, \tau, \kappa) &\in \mathcal{K} \times \mathcal{K}^* \times \mathbb{R}^m \times \mathbb{R}_+ \times \mathbb{R}_+. \end{aligned}$$

Observations

- Complementarity in **H**(\mathcal{K}) by definition: $x^T s = \tau \kappa = 0$.
- $\tau > 0 \Rightarrow$ Complementary solution for **P**(\mathcal{K}) and **D**(\mathcal{K}).
- $\kappa > 0 \Rightarrow$ Improving ray for **P**(\mathcal{K}) or **D**(\mathcal{K}).

The self-dual embedding



Third case: $\tau = \kappa = 0$

In Permenter, Friberg, and Andersen, *Solving conic optimization problems via self-dual embedding and facial reduction: a unified approach*, 2015:

$$\begin{aligned} Ax &= 0, \\ A^T y + s &= 0, \\ b^T y - c^T x &= 0, \\ (x, s, y, \tau, \kappa) &\in \mathcal{K} \times \mathcal{K}^* \times \mathbb{R}^m \times 0 \times 0. \end{aligned}$$



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$$\begin{aligned}Ax &= 0, \\A^T y + s &= 0, \\b^T y - c^T x &= 0, \\(x, s, y, \tau, \kappa) &\in \mathcal{K} \times \mathcal{K}^* \times \mathbb{R}^m \times 0 \times 0.\end{aligned}$$

Points like $(x, s, y, \tau, \kappa) = (0, 0, 0, 0, 0)$ will never be meaningful!



The self-dual embedding



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$$\begin{aligned}Ax &= 0, \\A^T y + s &= 0, \\b^T y - c^T x &= 0, \\(x, s, y, \tau, \kappa) &\in \mathcal{K} \times \mathcal{K}^* \times \mathbb{R}^m \times 0 \times 0.\end{aligned}$$

Key note from Theorem 1

Considering relint $\mathbf{H}(\mathcal{K})$, then $\tau = \kappa = 0$ is exceptional.

- If $\mathbf{P}(\mathcal{K})$ is troublesome: $\mathcal{K} \rightarrow \mathcal{K} \cap s^\perp$ (facial reduction).
- If $\mathbf{D}(\mathcal{K})$ is troublesome: $\mathcal{K}^* \rightarrow \mathcal{K}^* \cap x^\perp$ (facial reduction).



The self-dual embedding method
can solve all instances!

Assuming oracle for relint $\mathbf{H}(\mathcal{K})$ points.



The MOSEK Conic Optimizer



An oracle for relint $\mathbf{H}(\mathcal{K})$ points?

In theory:

Yes, if tracking the central path with infinite precision.



The MOSEK Conic Optimizer



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In theory:

Yes, if tracking the central path with infinite precision.

In practice:

Issues in exceptional $\tau = \kappa = 0$ case:

- Lack of strict complementarity
⇒ **slower and less accurate convergence.**
- Ill-posedness (i.e., instability to infinitesimal perturbations)
⇒ **overlapping termination criteria.**
- Interior-point method
⇒ **when are τ and κ small enough?**
⇒ **have to be careful about orthogonal complements,**
e.g., when computing $\mathcal{K} \cap s^\perp$.





Facial reduction certificate of $\mathbf{D}(\mathcal{K})$

In theory:

$$c^T x = 0, \quad Ax = 0, \quad x \in \mathcal{K} \setminus (\mathcal{K}^*)^\perp.$$



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$$c^T x = 0, \quad Ax = 0, \quad x \in \mathcal{K} \setminus (\mathcal{K}^*)^\perp.$$

This implies $x^T(c - A^T y) = x^T s \iff x^T s = 0$ and thus

$$s \in \mathcal{K}^* \longrightarrow s \in \mathcal{K}^* \cap x^\perp$$



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$$c^T x = 0, \quad Ax = 0, \quad x \in \mathcal{K} \setminus (\mathcal{K}^*)^\perp.$$

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$$s \in \mathcal{K}^* \longrightarrow s \in \mathcal{K}^* \cap x^\perp \text{ (a proper face of } \mathcal{K}^*)$$



Facial reduction certificate of $\mathbf{D}(\mathcal{K})$

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$$c^T x = 0, \quad Ax = 0, \quad x \in \mathcal{K} \setminus (\mathcal{K}^*)^\perp.$$

This implies $x^T(c - A^T y) = x^T s \iff x^T s = 0$ and thus

$$s \in \mathcal{K}^* \longrightarrow s \in \mathcal{K}^* \cap x^\perp \text{ (a proper face of } \mathcal{K}^*)$$

In practice:

- Have $x \in \mathcal{K}$.
- Need negligible $|c^T x|$ and $\|Ax\|$.
- Need significant $\text{dist}(x, (\mathcal{K}^*)^\perp)$. ($= \|x\|$ for proper cones)



Facial reduction certificate of $\mathbf{P}(\mathcal{K})$

In theory:

$$b^T y = 0, \quad A^T y + s = 0, \quad s \in \mathcal{K}^* \setminus \mathcal{K}^\perp$$

This implies $y^T A x = y^T b \iff s^T x = 0$ and thus

$$x \in \mathcal{K} \longrightarrow x \in \mathcal{K} \cap s^\perp \text{ (a proper face of } \mathcal{K}\text{)}$$

In practice:

- Have $s \in \mathcal{K}^*$.
- Need negligible $|b^T y|$ and $\|A^T y + s\|$.
- Need significant $\text{dist}(s, (\mathcal{K}^*)^\perp)$. (= $\|s\|$ for proper cones)



Facial reduction certificate of $\mathbf{P}(\mathcal{K})$

In theory:

$$b^T y = 0, \quad A^T y + s = 0, \quad s \in \mathcal{K}^* \setminus \mathcal{K}^\perp$$

This implies $y^T A x = y^T b \iff s^T x = 0$ and thus

$$x \in \mathcal{K} \longrightarrow x \in \mathcal{K} \cap s^\perp \text{ (a proper face of } \mathcal{K}\text{)}$$

In practice:

- Have $s \in \mathcal{K}^*$.
- Need negligible $|b^T y|$ and $\|A^T y + s\|$.
- Need significant $\text{dist}(s, (\mathcal{K}^*)^\perp)$. ($\approx \|y\|$ significant)

The MOSEK Conic Optimizer

Termination criteria for ill-posedness



Matrix A needs to be well-scaled and of full row rank.



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If not satisfied, we may get $\|A^T y + s\| = 0$ for significant $\|y\|$ and negligible $\|s\|$.





Matrix A needs to be well-scaled and of full row rank.

If not satisfied, we may get $\|A^T y + s\| = 0$ for significant $\|y\|$ and negligible $\|s\|$. But then:

Linear row dependency certificate of $\mathbf{P}(\mathcal{K})$

$$y^T A = 0, \quad y^T b = 0, \quad y \neq 0$$





Our proposal

In principle:

- 1 Negligible: $\|Ax\|$ **and** $|c^T x|$.
- 2 Negligible: $\|A^T y + s\|$ **and** $|b^T y|$.
- 3 Significant: $\|x\|$ **or** $\|y\|$.

In implementation:

- 1 $\max(\|Ax\|, |c^T x|) \leq \epsilon \max(1, \|x\|)$.
- 2 $\max(\|A^T y + s\|, |b^T y|) \leq \epsilon \max(1, \|y\|)$.
- 3 $\max(\|Ax\|, |c^T x|) < \epsilon \|x\|$ **or**
 $\max(\|A^T y + s\|, |b^T y|) < \epsilon \|y\|$.



H. Waki, M. Nakata and M. Muramatsu, "Strange Behaviors of Interior-point Methods for Solving Semidefinite Programming Problems in Polynomial Optimization", 2012:

unboundDim1R*

- Dual is SDP-relaxation of POP (minimized).
- Primal ill-posed with $(p;d) = (0.0; 0.0^+)$.





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*	Status
1	Complementary solution of value 0.0
2	Near to complementary solution of value 0.0
3-4	Ill-posed certificate
5-10	Complementary solution of value 1.0



The MOSEK Conic Optimizer



unboundDim1R1

ITE	PFEAS	DFEAS	GFEAS	MU
0	1.0e+00	1.0e+00	2.0e+00	1.0e+00
1	3.1e-02	3.1e-02	1.1e+00	3.1e-02
2	4.8e-05	4.8e-05	2.4e-02	4.8e-05
3	1.1e-09	1.1e-09	2.2e-09	1.1e-09

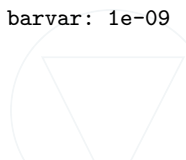
Interior-point optimizer terminated. Time: 0.00.

Problem status : PRIMAL_AND_DUAL_FEASIBLE

Solution status : OPTIMAL

Primal. obj: -1.3336443272e-09 nrm: 1e+00 con: 1e-09 var: 0e+00
barvar: 2e-20

Dual. obj: 7.0491750159e-10 nrm: 2e+00 con: 0e+00 var: 1e-09
barvar: 1e-09



The MOSEK Conic Optimizer



unboundDim1R²

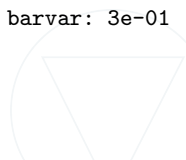
ITE	PFEAS	DFEAS	GFEAS	MU
0	2.0e+00	2.0e+00	1.0e+00	1.0e+00
1	2.9e-01	2.9e-01	1.7e-01	1.5e-01
2	5.2e-02	5.2e-02	7.2e-02	2.6e-02
...				
37	4.1e-21	5.4e-20	4.7e-19	2.0e-21
38	4.1e-21	5.4e-20	4.7e-19	2.0e-21
39	4.1e-21	5.4e-20	4.7e-19	2.0e-21

Interior-point optimizer terminated. Time: 0.01.

Problem status : PRIMAL_AND_DUAL_FEASIBLE

Solution status : NEAR_OPTIMAL

Primal.	obj: 2.1214425554e-04	nrm: 1e+00	con: 3e-13	barvar: 0e+00
Dual.	obj: 1.9285790417e-04	nrm: 3e+08	con: 0e+00	barvar: 3e-01



The MOSEK Conic Optimizer



unboundDim1R² (note: overlapping termination criteria)

ITE	PFEAS	DFEAS	GFEAS	MU
0	2.0e+00	2.0e+00	1.0e+00	1.0e+00
1	2.9e-01	2.9e-01	1.7e-01	1.5e-01
2	5.2e-02	5.2e-02	7.2e-02	2.6e-02
...				
37	4.1e-21	5.4e-20	4.7e-19	2.0e-21
38	4.1e-21	5.4e-20	4.7e-19	2.0e-21
39	4.1e-21	5.4e-20	4.7e-19	2.0e-21

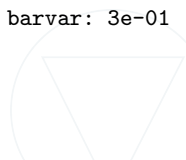
Interior-point optimizer terminated. Time: 0.01.

Problem status : PRIMAL_AND_DUAL_FEASIBLE

Solution status : NEAR_OPTIMAL

Primal. obj: 2.1214425554e-04 nrm: 1e+00 con: 3e-13 barvar: 0e+00

Dual. obj: 1.9285790417e-04 **nrm: 3e+08** con: 0e+00 barvar: 3e-01





ITE	PFEAS	DFEAS	GFEAS	MU
0	2.0e+00	2.0e+00	1.0e+00	1.0e+00
1	3.3e-01	3.3e-01	1.5e-01	1.7e-01
2	5.5e-02	5.5e-02	6.7e-02	2.8e-02
...				
30	7.9e-17	2.7e-15	6.6e-16	4.0e-17
31	7.9e-17	2.7e-15	6.6e-16	4.0e-17
32	7.9e-17	2.7e-15	6.6e-16	4.0e-17

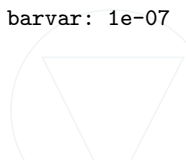
Interior-point optimizer terminated. Time: 0.01.

Interior-point solution summary

Problem status : ILL_POSED

Solution status : PRIMAL_ILLPOSED_CER

Dual. obj: 1.3716138286e-08 nrm: 5e+00 con: 0e+00 barvar: 1e-07





H. Waki: "How to generate weakly infeasible semidefinite programs via Lasserre's relaxations for polynomial optimization", 2012:

CompactDim2R*

- Dual is SDP-relaxation of POP (minimized).
- Primal ill-posed with $(p;d) = (-\infty; -\infty^+)$.





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CompactDim2R*

- Dual is SDP-relaxation of POP (minimized).
- Primal ill-posed with $(p;d) = (-\infty; -\infty^+)$.
- Higher-order Lasserre hierarchy means smaller perturbation exist after which $(p;d) = (-1.5; -1.5)$.

*	Status
1	Primal infeasibility certificate
2	Ill-posed certificate
3-10	Complementary solution of value -1.5



The MOSEK Conic Optimizer



CompactDim2R¹

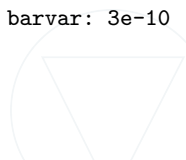
ITE	PFEAS	DFEAS	GFEAS	MU
0	2.0e+00	1.0e+00	2.0e+00	1.0e+00
1	4.2e-01	2.1e-01	4.2e-01	2.1e-01
2	7.1e-02	3.6e-02	5.3e-02	3.6e-02
...				
23	1.2e-13	8.0e-14	4.9e-15	6.2e-14
24	2.6e-14	1.6e-14	3.1e-16	1.3e-14
25	3.5e-15	3.1e-15	5.3e-17	1.8e-15

Interior-point optimizer terminated. Time: 0.00.

Problem status : PRIMAL_INFEASIBLE

Solution status : PRIMAL_INFEASIBLE_CER

Dual. obj: -4.2224567689e-05 nrm: 3e+00 con: 0e+00 var: 2e-10
barvar: 3e-10



The MOSEK Conic Optimizer



CompactDim2R¹ (note: overlapping termination criteria)

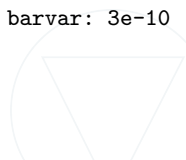
ITE	PFEAS	DFEAS	GFEAS	MU
0	2.0e+00	1.0e+00	2.0e+00	1.0e+00
1	4.2e-01	2.1e-01	4.2e-01	2.1e-01
2	7.1e-02	3.6e-02	5.3e-02	3.6e-02
...				
23	1.2e-13	8.0e-14	4.9e-15	6.2e-14
24	2.6e-14	1.6e-14	3.1e-16	1.3e-14
25	3.5e-15	3.1e-15	5.3e-17	1.8e-15

Interior-point optimizer terminated. Time: 0.00.

Problem status : PRIMAL_INFEASIBLE

Solution status : PRIMAL_INFEASIBLE_CER

Dual. obj: **-4.2224567689e-05** nrm: 3e+00 con: 0e+00 var: 2e-10
barvar: 3e-10





ITE	PFEAS	DFEAS	GFEAS	MU
0	2.0e+00	1.5e+00	1.5e+00	1.0e+00
1	4.4e-01	3.3e-01	3.8e-01	2.2e-01
2	9.2e-02	6.9e-02	2.3e-01	4.6e-02
...				
25	1.8e-15	3.6e-15	5.0e-16	9.0e-16
26	1.8e-15	3.6e-15	5.0e-16	9.0e-16
27	1.8e-15	3.6e-15	5.0e-16	9.0e-16

Interior-point optimizer terminated. Time: 0.01.

Problem status : ILL_POSED

Solution status : PRIMAL_ILLPOSED_CER

Dual. obj: -1.7098776698e-06 nrm: 7e+00 con: 0e+00 barvar: 1e-08





$$\text{maximize} \quad \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \bar{X} \right\rangle - 0.5x_0 + 0.5x_1 + 0.5x_2$$

$$\text{subject to} \quad \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \bar{X} \right\rangle - x_1 = 1.0$$

$$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \bar{X} \right\rangle - x_2 = 1.0$$

$$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{X} \right\rangle = 0.0$$

$$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{X} \right\rangle + x_0 = 0.0$$

$$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{X} \right\rangle = 0.0$$

$$\bar{X} \in \mathcal{S}_+^3$$

$$x \in \mathbb{R}_+^3$$





$$\text{maximize } \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \bar{X} \right\rangle - 0.5x_0 + 0.5x_1 + 0.5x_2$$

$$\text{subject to } \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \bar{X} \right\rangle - x_1 = 1.0 \quad [10^{-5}]$$

$$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \bar{X} \right\rangle - x_2 = 1.0 \quad [10^{-5}]$$

$$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{X} \right\rangle = 0.0 \quad [3]$$

$$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{X} \right\rangle + x_0 = 0.0 \quad [-0.4]$$

$$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{X} \right\rangle = 0.0 \quad [3]$$

$$\bar{X} \in \mathcal{S}_+^3 \begin{bmatrix} -3 & & \\ 0.4 & -3 & \\ 10^{-5} & 10^{-5} & 10^{-10} \end{bmatrix} \quad x \in \mathbb{R}_+^3 \begin{bmatrix} 0.4 \\ 10^{-5} \\ 10^{-5} \end{bmatrix}$$



$$\text{maximize} \quad \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \bar{X} \right\rangle - 0.5x_0 + 0.5x_1 + 0.5x_2$$

$$\text{subject to} \quad \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \bar{X} \right\rangle - x_1 = 1.0 \quad [10^{-5}]$$

$$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \bar{X} \right\rangle - x_2 = 1.0 \quad [10^{-5}]$$

$$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{X} \right\rangle = 0.0 \quad [3]$$

$$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{X} \right\rangle + x_0 = 0.0 \quad [-0.4]$$

$$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{X} \right\rangle = 0.0 \quad [3]$$

$$\bar{X} \in \mathcal{S}_+^3 \begin{bmatrix} -3 & & \\ 0.4 & -3 & \\ 10^{-5} & 10^{-5} & 10^{-10} \end{bmatrix} \quad x \in \mathbb{R}_+^3 \begin{bmatrix} 0.4 \\ 10^{-5} \\ 10^{-5} \end{bmatrix}$$



$$\text{maximize} \quad \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \bar{X} \right\rangle - 0.5x_0 + 0.5x_1 + 0.5x_2$$

$$\text{subject to} \quad \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \bar{X} \right\rangle - x_1 = 1.0 \quad [10^{-5}]$$

$$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \bar{X} \right\rangle - x_2 = 1.0 \quad [10^{-5}]$$

$$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{X} \right\rangle = 0.0 \quad [3]$$

$$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{X} \right\rangle + x_0 = 0.0 \quad [-0.4]$$

$$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{X} \right\rangle = 0.0 \quad [3]$$

$$\bar{X} \in \mathcal{S}_+^3 \begin{bmatrix} -3 & & \\ 0.4 & -3 & \\ 10^{-5} & 10^{-5} & 10^{-10} \end{bmatrix} \quad x \in \mathbb{R}_+^3 \begin{bmatrix} 0.4 \\ 10^{-5} \\ 10^{-5} \end{bmatrix}$$



Liu and Pataki, *“Exact duals and short certificates of infeasibility and weak infeasibility in conic linear programming”*, 2015:

n=10, m=10	Strongly/weakly infeasible mix		Weakly infeasible	
	Clean	Messy	Clean	Messy
SEDUMI	87	27	0	0
SDPT3	10	5	0	0
MOSEK 7-series	63	17	0	0
MOSEK 8.0.0.33(BETA)	98w89	65w64	100w100	0
PP+SEDUMI	100	27	100	0





Liu and Pataki, “Exact duals and short certificates of infeasibility and weak infeasibility in conic linear programming”, 2015:

n=10, m=20	Strongly/weakly infeasible mix		Weakly infeasible	
	Clean	Messy	Clean	Messy
SEDUMI	100	100	1	0
SDPT3	100	96	0	0
MOSEK 7-series	100	100	11	0
MOSEK 8.0.0.33(BETA)	100w1	69w8	98w98	0
PP+SEDUMI	0	100	0	0

