Disjunctive conic cuts: The good, the bad, and implementation

MOSEK workshop on Mixed-integer conic optimization

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Motivation

Goals!

- Extend the ideas of disjunctive programming to quadratic problems.
- Derive disjunctive conic cuts for MISOCO:
  - Solve the continuous relaxation (a SOCO problem).
  - Identify a violated disjunction (fractional variable).
  - Design a cut to approximate convex hull of disjunctive set.
  - Include cuts in branch and cut algorithm.
- Show that these ideas may be used to derive valid inequalities for some non-convex quadratic sets.

Mixed integer second order cone optimization (MISOCO)

$$
\begin{aligned}
\text{min:}\ & c^T x \\
\text{s.t.:}\ & Ax = b \qquad\qquad \text{(MISOCO)} \\
& x \in \mathbb{L}^n \\
& x \in \mathbb{Z}^d \times \mathbb{R}^{n-d},
\end{aligned}
$$

where:

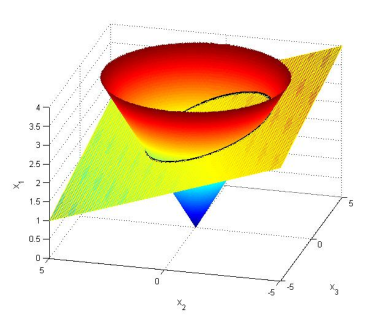
- $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$,
- $\mathbb{L}^n = \{x \in \mathbb{R}^n | x_1 \geq \|x_{2:n}\|\}$,
- Rows of A are linearly independent.
- Here $\|x\|$ denotes Euclidean norm of x.

Continuous relaxation

$$\text{min:}\quad x_1 - 2x_2 + x_3$$

$$\text{s.t.:}\quad x_1 - 0.1x_2 + 0.2x_3 = 2.5$$

$$x_1 \geq \left\| \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \right\|$$



Feasible set

MISOCO example: Solve the relaxed problem

Find the optimal solution x^*_{soco} for the continuous relaxation

$$\begin{array}{rrrrrl} \text{min:} & 3x_1 & +2x_2 & +2x_3 & +x_4 & \\ \text{s.t.:} & 9x_1 & +x_2 & +x_3 & +x_4 & = 10 \\ & & & (x_1, x_2, x_3, x_4) & \in \mathbb{L}^4 & \\ & & & x_4 & \in \mathbb{Z}. & \end{array}$$

Relaxing the integrality constraint we get the optimal solution:

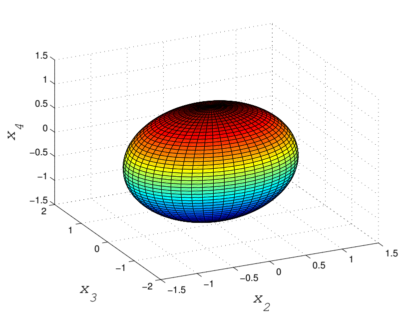
$$x^*_{\text{soco}} = (1.36, -0.91, -0.91, -0.45),$$

with and optimal objective value: $z^* = 0.00$.

MISOCO example: Reformulation

Reformulation of the relaxed problem

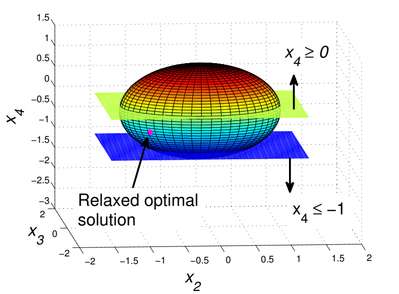
$$\begin{array}{lll} \text{min:} & \frac{1}{3}\left(10 + 5x_2 + 5x_3 + 2x_4\right) & \\ \text{s.t.:} & \begin{bmatrix} x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 8 & -\frac{1}{10} & -\frac{1}{10} \\ -\frac{1}{10} & 8 & -\frac{1}{10} \\ -\frac{1}{10} & -\frac{1}{10} & 8 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} - 10 & \leq 0 \\ & x_4 \in \mathbb{Z}. & \end{array}$$



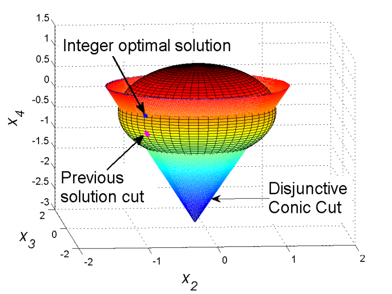
Feasible set of the reformulated problem

MISOCO example: Find a violated disjunction & cut

The disjunction $x_4 \leq -1 \bigvee x_4 \geq 0$ is violated by x^*_{soco}



(A) Disjunction



(B) Disjunctive conic cut

An integer optimal solution is obtained after adding one cut:

$$x^*_{\text{misoco}} = x^*_{\text{soco}} = (1.32,\ -0.93,\ -0.93,\ 0.00),$$

with an optimal objective value: $z^*_{\text{misoco}} = z^*_{\text{soco}} = 0.24.$

Applications

- Computer vision and pattern recognition
    - Kumar, Torr, and Zisserman (2006).
- Portfolio optimization with round lot purchasing constraints
    - Bonami and Lejeune (2009)
- Location-inventory problems
    - Atamtürk, Berenguer, and Shen (2009)
- Joint network optimization and beamforming
    - Cheng, Drewes, Philipp, and Pesavento (2012)
- Infrastructure planning for electric vehicles
    - Mak, Rong, and Shen (2013)
- Sequencing appointments for service systems
    - Mak, Rong, and Zhang (2014)
- The design of service systems with congestion
    - Góez and Anjos (2017)

Related work

- A Complete Characterization of Disjunctive Conic Cuts for Mixed Integer Second Order Cone Optimization
    - Belotti, Góez, Pólik, Ralphs, Terlaky (2017).
- Intersection cuts for nonlinear integer programming: Convexification techniques for structured sets
    - Modaresi, Kılınç, Vielma (2016).
- Two-term disjunctions on the second-order cone
    - Kılınç-Karzan, Yıldız (2015).
- Disjunctive cuts for cross-sections of the second-order cone
    - Yıldız, Cornuéjols (2015).

The good

Quadratic sets and disjunctions

$$\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^T P x + 2p^T x + \rho \leq 0\}$$
$$\mathcal{A} = \{x \in \mathbb{R}^n \mid a^\top x \geq \alpha\}$$
$$\mathcal{B} = \{x \in \mathbb{R}^n \mid b^\top x \leq \beta\}$$

- $P \in \mathbb{R}^{n \times n}$ with $n-1$ positive and exactly one non-positive eigenvalues, $p \in \mathbb{R}^n$, $\rho \in \mathbb{R}$.
- $\|a\| = \|b\| = 1$ and $\beta < \alpha$.
- We denote the boundaries of the half-spaces $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A}^=$ and $\mathcal{B}^=$ respectively.
- The intersection $\mathcal{Q} \cap \mathcal{A} \cap \mathcal{B}$ results in the disjunctive sets $\mathcal{Q} \cap \mathcal{A}$ and $\mathcal{Q} \cap \mathcal{B}$
- We assume that $\mathcal{A}^= \cap \mathcal{Q} \neq \emptyset$, $\mathcal{B}^= \cap \mathcal{Q} \neq \emptyset$, and $\mathcal{B}^= \cap \mathcal{A}^= \cap \mathcal{Q} = \emptyset$.

A family of quadratic inequalities

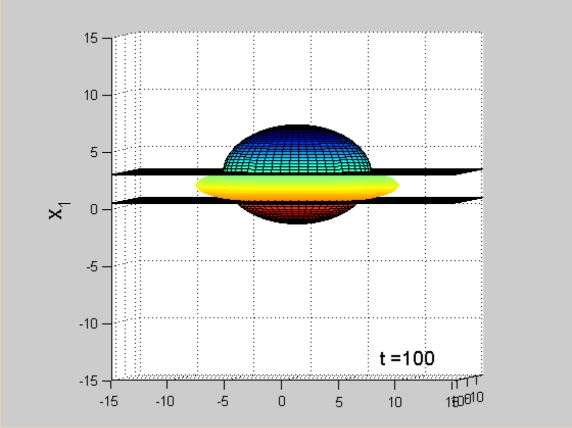
Let $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ be a family of quadratic sets having the same intersection with $\mathcal{A}^=$ and $\mathcal{B}^=$, with

$$\begin{aligned}\mathcal{Q}(\tau) &= \left\{x \in \mathbb{R}^n \mid \left(x^\top P x + 2p^\top x + \rho\right) + \tau \left(a^\top x - \alpha\right)\left(b^\top x - \beta\right) \leq 0\right\}\\ &= \{x \in \mathbb{R}^n \mid x^\top P(\tau) x + 2p(\tau)^\top x + \rho(\tau) \leq 0\}.\end{aligned}$$

where

- $P(\tau) = P + \tau \frac{ab^\top + ba^\top}{2}$
- $p(\tau) = p - \tau \frac{\beta a + \alpha b}{2}$
- $\rho(\tau) = \rho + \tau \alpha\beta$

A family of quadratic inequalities



Sequence of quadrics $w^\top P(\tau)w + 2p(\tau)^\top w + \rho(\tau) \le 0$, for $-101 \le \tau \le 100$.

Family of quadrics intersecting two parallel hyperplanes

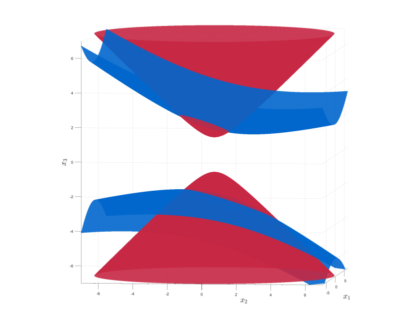
Range	$(P(\tau), p(\tau), \rho(\tau))$
$\tau > -8.9875$	*Ellipsoids*
$\hat{\tau} = -8.9875$	*Paraboloid*
$-9.5903 < \tau < -8.9875$	*Two sheets hyperboloids*
$\bar{\tau}_2 = -9.5903$	*Cone*
$-101.7697 < \tau < -9.5903$	*One sheet hyperboloids*
$\bar{\tau}_1 = -101.7697$	*Cone*
$\tau < -101.7697$	*Two sheets hyperboloids*

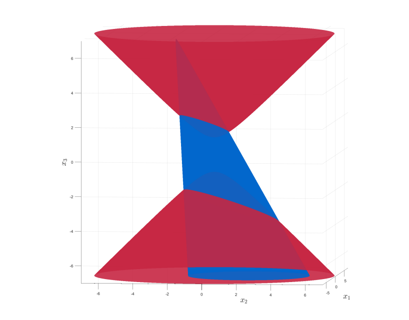
Behavior of the quadrics for different ranges of τ.

A family of valid quadratic inequalities

Corollary

Given a quadratic set $\mathcal{Q}$ and two half spaces $\mathcal{A}$ and $\mathcal{B}$, any quadratic set in the family $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ is a valid quadratic inequality for $\mathcal{Q} \cap (\mathcal{A}^{=} \cup \mathcal{B}^{=})$.

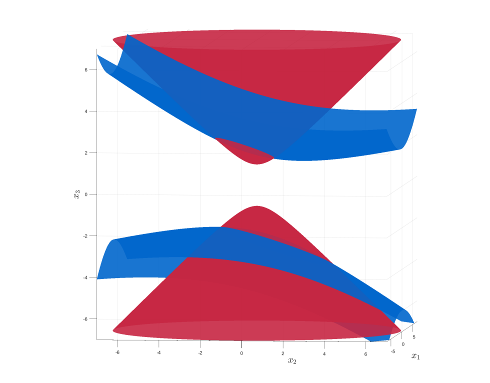




A family of valid quadratic inequalities

Theorem

Given a quadratic set $\mathcal{Q}$ *and two half spaces* $\mathcal{A}$ *and* $\mathcal{B}$ *such that* $\mathcal{B}^{=} \cap \mathcal{A}^{=} \cap \mathcal{Q} = \emptyset$, *a quadratic set in the family* $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ *is a valid quadratic inequality for* $\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$ *if and only if* $\tau \leq 0$.

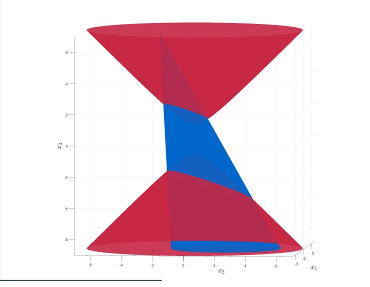


A family of valid quadratic inequalities

Theorem

Consider a quadratic set $\mathcal{Q}$ and two half-spaces $\mathcal{A}$ and $\mathcal{B}$. If there exists a $\bar{\tau}$ such that $\mathcal{Q}(\bar{\tau}) = \mathcal{Q}_1(\bar{\tau}) \cup \mathcal{Q}_2(\bar{\tau})$ is a non-convex quadratic cone, and its vertex v is contained in $\mathcal{A}$ or $\mathcal{B}$ but not in $\mathcal{A} \cap \mathcal{B}$, then each branch $i = 1, 2$ of $\mathcal{Q}(\bar{\tau})$ is a valid quadratic inequality for $\mathcal{Q} \cap (\mathcal{A}_i^{=}(\bar{\tau}) \cup \mathcal{B}_i^{=}(\bar{\tau}))$[1], such that

$$\operatorname{conv}(\mathcal{Q} \cap (\mathcal{A}_i^{=}(\bar{\tau}) \cup \mathcal{B}_i^{=}(\bar{\tau}))) = \operatorname{conv}(\mathcal{Q}_i(\bar{\tau}) \cap (\mathcal{A}_i^{=}(\bar{\tau}) \cup \mathcal{B}_i^{=}(\bar{\tau}))) \subseteq \mathcal{Q}_i(\bar{\tau}).$$



[1] $\mathcal{A}_1^{=}(\bar{\tau}) = \mathcal{A}^{=} \cap \mathcal{Q}_1(\bar{\tau})$, $\mathcal{A}_2^{=}(\bar{\tau}) = \mathcal{A}^{=} \cap \mathcal{Q}_2(\bar{\tau})$, and similarly we define $\mathcal{B}_1^{=}(\bar{\tau})$, $\mathcal{B}_2^{=}(\bar{\tau})$

Parallel hyperplanes

Let $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ be a family of quadratic sets having the same intersection with $\mathcal{A}^=$ and $\mathcal{B}^=$, with

$$\begin{aligned}\mathcal{Q}(\tau) &= \left\{x \in \mathbb{R}^n \mid \left(x^\top Px + 2p^\top x + \rho\right) + \tau \left(a^\top x - \alpha\right)\left(a^\top x - \beta\right) \leq 0\right\} \\ &= \{x \in \mathbb{R}^n \mid x^\top P(\tau)x + 2p(\tau)^\top x + \rho(\tau) \leq 0\}\end{aligned}$$

where

- $P(\tau) = P + \tau aa^\top$
- $p(\tau) = p - \tau \frac{\beta+\alpha}{2} a$
- $\rho(\tau) = \rho + \tau\alpha\beta$

Non-convex quadratic cones in the family

Rewrite $Q(\tau)$ as:

$$\mathcal{Q}(\tau) = \{x \in \mathbb{R}^n \mid \left(x + P^{-1}(\tau)p(\tau)\right) P(\tau) \left(x + P^{-1}(\tau)p(\tau)\right) \\ \leq p(\tau)P^{-1}(\tau)p(\tau) - \rho(\tau)\}$$

and we obtain

$$p(\tau)^\top P(\tau)^{-1}p(\tau) - \rho(\tau) = \frac{\left(1-2a_1^2\right)\frac{(\alpha-\beta)^2}{4}\tau^2 - (\rho(1-2a_1^2)+\alpha\beta)\tau - \rho}{1+\tau(1-2a_1^2)}.$$

The roots $\bar{\tau}_1 \leq \bar{\tau}_2$ of the numerator are

$$2\left(\frac{\rho(1-2a_1^2)+\alpha\beta \pm \sqrt{(\rho(1-2a_1^2)+\beta^2)(\rho(1-2a_1^2)+\alpha^2)}}{(1-2a_1^2)(\alpha-\beta)^2}\right)$$

Non-convex quadratic cones in the family

Lemma

Let $\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2$ be one of the quadratic sets in our list. We have the following cases:

- *If $a_1^2 > \frac{1}{2}$ and the set $Q(\bar{\tau}_2)$ is a non-convex quadratic cone, then its vertex v is either in $\mathcal{A}$ or $\mathcal{B}$.*
- *If $a_1^2 = \frac{1}{2}$ and $\alpha\beta \geq 0$, then the set $Q(-\frac{\rho}{\alpha\beta})$ is a non-convex quadratic cone with its vertex v is either in $\mathcal{A}$ or $\mathcal{B}$.*
- *If $a_1^2 < \frac{1}{2}$ and the set $Q(\bar{\tau}_1)$ is a non-convex quadratic cone, then its vertex v is either in $\mathcal{A}$ or $\mathcal{B}$.*

Intersection of an affine space and a second order cone

- Null space representation of the affine space $\mathcal{H} = \{x \in \mathbb{R}^n \mid Ax = b\}$

$$\mathcal{H} := \{x \in \mathbb{R}^n \mid x = x^0 + Hw,\ \forall w \in \mathbb{R}^\ell\},$$

  where $\ell = n - m$, $Ax^0 = b$, and $H \in \mathbb{R}^{n \times \ell}$ is a basis for $\mathrm{Null}(A)$.
- There exist a matrix $P \in \mathbb{R}^{\ell \times \ell}$, $p \in \mathbb{R}^\ell$, $\rho \in \mathbb{R}$, s.t.

$$\mathcal{F} = \mathcal{H} \cap \mathbb{L}^n = \{x \in \mathbb{R}^n \mid \exists w \in \mathcal{F}^{\mathcal{Q}}, x = x^0 + Hw\},$$

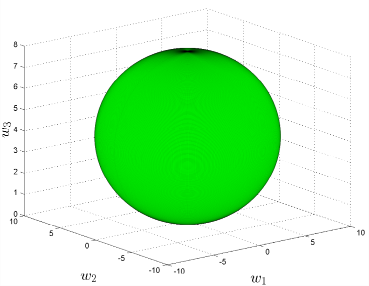
  where

$$\mathcal{F}^{\mathcal{Q}} = \{w \in \mathbb{R}^\ell \mid w^\top P w + 2p^\top w + \rho \leq 0,\ x_1^0 + H_{1:}^\top w \geq 0\}$$

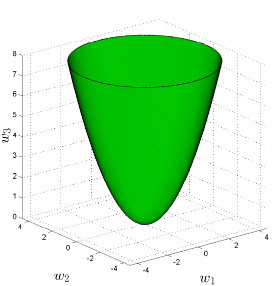
Theorem

The matrix P in the definition of the quadratic set $\color{red}{\mathcal{F}^{\mathcal{Q}}}$ has at most one non-positive eigenvalue, and at least $\ell - 1$ positive eigenvalues.

We only need to account for the following possible shapes for $\color{red}{\mathcal{F}^{\mathcal{Q}}}$:



Ellipsoid.



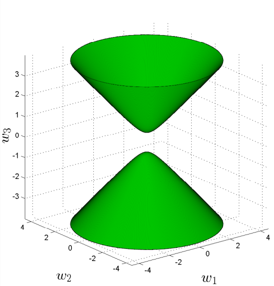
Paraboloid.

Intersection of an affine space and a second order cone

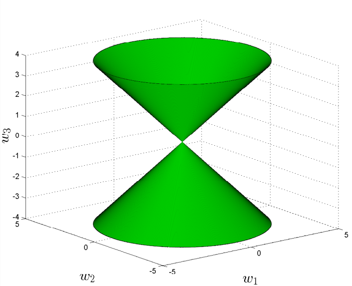
Theorem

The matrix P in the definition of the quadratic set $\mathcal{F}^{\mathcal{Q}}$ has at most one non-positive eigenvalue, and at least $\ell - 1$ positive eigenvalues.

We only need to account for the following possible shapes for $\mathcal{F}^{\mathcal{Q}}$:



One branch of a hyperboloid.



Second order cone.

Intersection of an affine space and a second order cone

Let $\mathcal{A} = \{w \in \mathbb{R}^\ell \mid a^\top w \geq \alpha\}$ and $\mathcal{B} = \{w \in \mathbb{R}^\ell \mid a^\top w \leq \beta\}$. Define $\mathcal{F}^D = \{x \in \mathbb{R}^n \mid \exists w \in \mathcal{F}^\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B}), x = x^0 + Hw\}$

Lemma

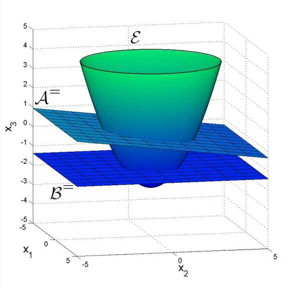
Given a vector $\hat{x} \in \mathcal{F}$ and a vector $\hat{w} \in \mathcal{F}^\mathcal{Q}$ such that $\hat{x} = x^0 + H\hat{w}$. Then $\hat{x} \notin \mathcal{F}^D$ if and only if $\hat{w} \notin \mathcal{F}^\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$.

Proof.

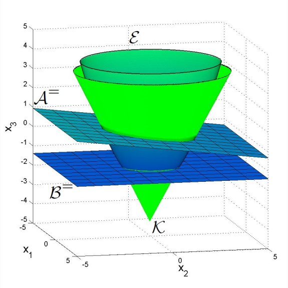
Note that any $x \in \mathcal{F}$ is a linear combination of x^0 and the columns of H. Additionally, recall that the columns of H are linearly independent. Then, the vector $\hat{w}$ defining $\hat{x}$ is unique. The result follows. $\square$

Disjunctive conic cuts: General theory

- Study the intersection of a convex set $\mathcal{E}$ and a disjunctive set $\mathcal{A} = \{x \in \mathbb{R}^n \mid a^\top x \geq \alpha\} \cup \mathcal{B} = \{x \in \mathbb{R}^n \mid b^\top x \leq \beta\}$[2].
- Show that under some mild assumptions $\text{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$ can be characterized using a convex cone $\mathcal{K}$.



(A)



(B)

[2] $\mathcal{A}^= = \{x \in \mathbb{R}^n \mid a^\top x = \alpha\}$ and $\mathcal{B}^= = \{x \in \mathbb{R}^n \mid b^\top x = \beta\}$

Disjunctive conic cuts (DCCs): Definition

Definition

A closed convex cone $\mathcal{K} \in \mathbb{R}^n$ with $\dim(\mathcal{K}) > 1$ is called a *Disjunctive Conic Cut* (DCC) for $\mathcal{E}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$ if

$$\operatorname{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{E} \cap \mathcal{K}.$$

Assumption

The intersection $\mathcal{A} \cap \mathcal{B} \cap \mathcal{E}$ is empty.

Assumption

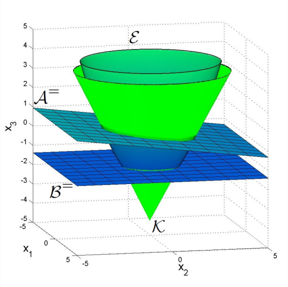
The intersections $\mathcal{E} \cap \mathcal{A}^{=}$ and $\mathcal{E} \cap \mathcal{B}^{=}$ are nonempty and bounded.

Disjunctive conic cuts: Characterization

Proposition

A closed convex cone $\mathcal{K} \in \mathbb{R}^n$ with $\dim(\mathcal{K}) > 1$ *is a DCC for $\mathcal{E}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$, if*

$$\mathcal{K} \cap \mathcal{A}^= = \mathcal{E} \cap \mathcal{A}^= \quad \textit{and} \quad \mathcal{K} \cap \mathcal{B}^= = \mathcal{E} \cap \mathcal{B}^=.$$



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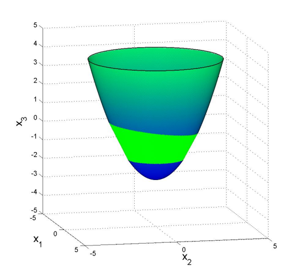
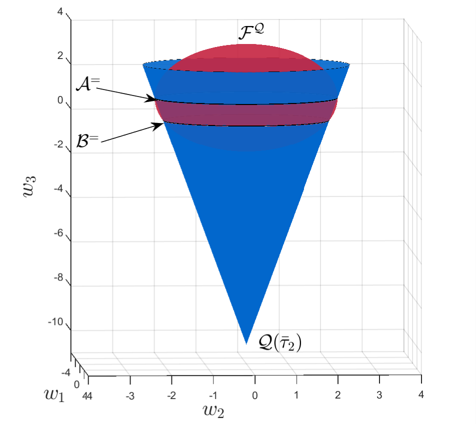


(B)

DCCs for MISOCO when intersections are bounded

Theorem

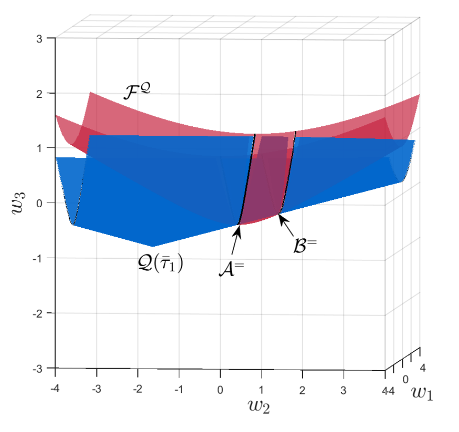
Let $\mathcal{A}^= = \{w \in \mathbb{R}^\ell | a^\top w = \alpha\}$ *and* $\mathcal{B}^= = \{w \in \mathbb{R}^\ell | a^\top w = \beta\}$ *be given. If the sets* $\mathcal{A}^= \cap \mathcal{F}^\mathcal{Q}$ *and* $\mathcal{B}^= \cap \mathcal{F}^\mathcal{Q}$ *are bounded, then the quadric* $\mathcal{Q}(\bar{\tau}_2)$ *contains a DCC for MISOCO.*



DCCs for MISOCO when intersections are unbounded

Theorem

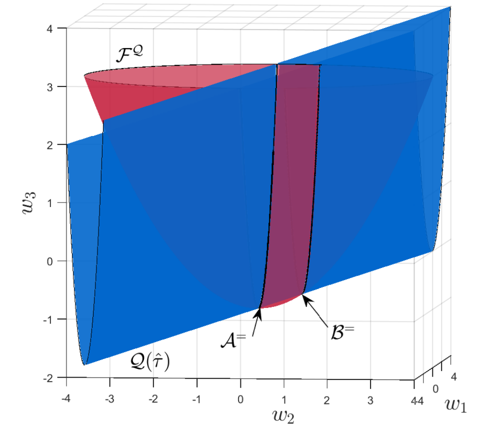
Let $\mathcal{A}^= = \{w \in \mathbb{R}^\ell | a^\top w = \alpha\}$ and $\mathcal{B}^= = \{w \in \mathbb{R}^\ell | a^\top w = \beta\}$ be given. If P is non-singular and the sets $\mathcal{A}^= \cap \mathcal{F}^\mathcal{Q}$ and $\mathcal{B}^= \cap \mathcal{F}^\mathcal{Q}$ are unbounded, then the quadric $\mathcal{Q}(\bar{\tau}_1)$ contains a DCC for MISOCO.



DCCs for MISOCO when intersections are unbounded

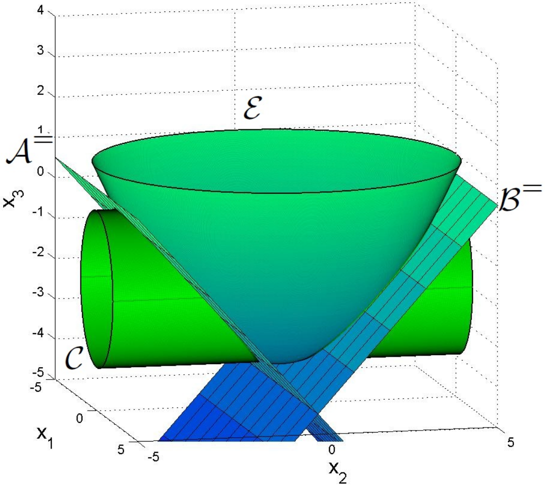
Theorem

Let $\mathcal{A}^= = \{w \in \mathbb{R}^\ell \mid a^\top w = \alpha\}$ and $\mathcal{B}^= = \{w \in \mathbb{R}^\ell \mid a^\top w = \beta\}$ be given. If P is singular and the sets $\mathcal{A}^= \cap \mathcal{F}^\mathcal{Q}$ and $\mathcal{B}^= \cap \mathcal{F}^\mathcal{Q}$ are unbounded, then the quadric $\mathcal{Q}(\hat{\tau})$ is a DCC for MISOCO.



What happens if the hyperplanes are non-parallel?

The results for the bounded intersections still hold.



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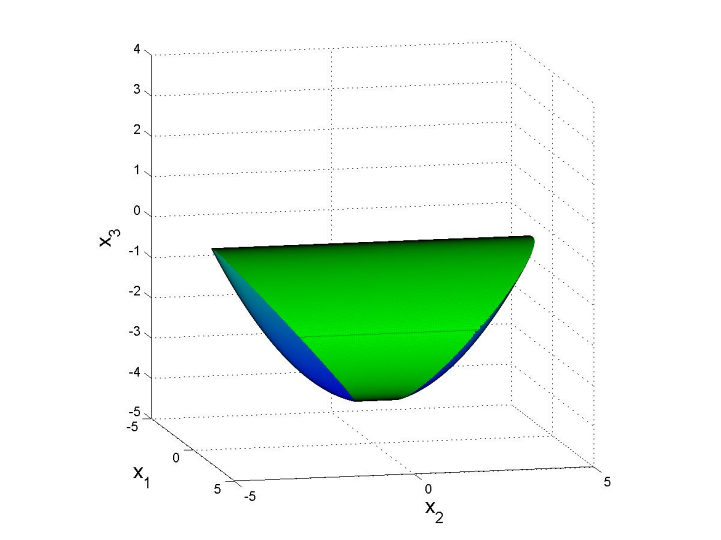
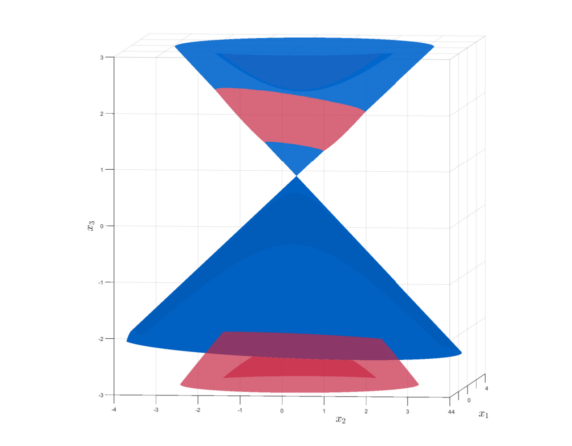


Does this approach work beyond MISOCO?

Let us consider

- Hyperboloids of two sheets and non-convex quadratic cones.
- Hyperboloids of one sheet.

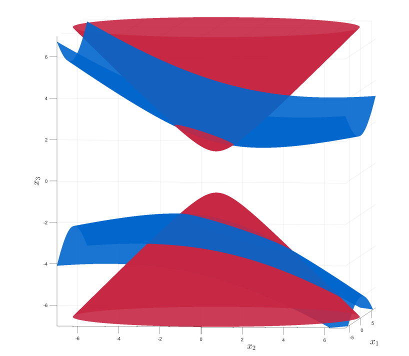
The sets $\mathcal{Q} \cap \mathcal{A}^{=}$ and $\mathcal{Q} \cap \mathcal{B}^{=}$ are bounded, $a_1 > \frac{1}{2}$



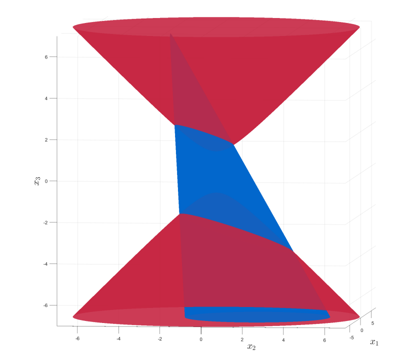
Valid conic inequality $\mathcal{Q}(\bar{\tau}_2)$ when both hyperplanes intersecting the same branch of $\mathcal{Q}$

The sets $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are bounded, $a_1 > \frac{1}{2}$

Both hyperplanes intersecting different branches of $\mathcal{Q}$

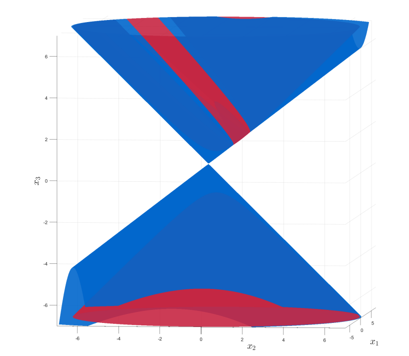


Valid conic inequality $\mathcal{Q}(\bar{\tau}_2)$



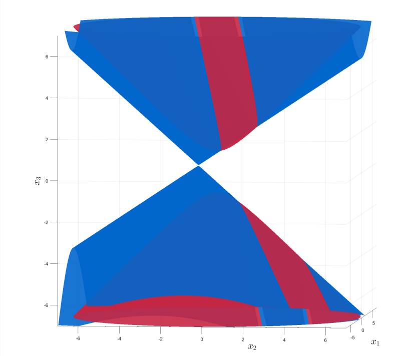
Valid conic inequality $\mathcal{Q}(\bar{\tau}_1)$

The sets $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are unbounded, $a_1 = \frac{1}{2}$



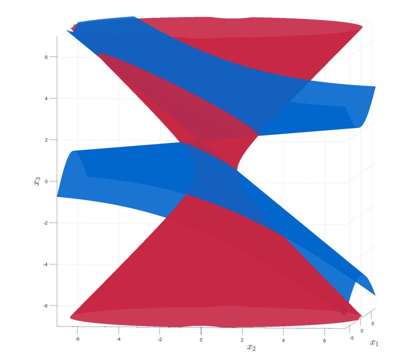
Valid conic inequality $\mathcal{Q}(\bar{\tau}_2)$, in this case $\text{conv}(\mathcal{Q}_1 \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{Q}_1 \cap \mathcal{Q}(\bar{\tau})$

The sets $\mathcal{Q} \cap \mathcal{A}^{=}$ and $\mathcal{Q} \cap \mathcal{B}^{=}$ are unbounded, $a_1 < \frac{1}{2}$

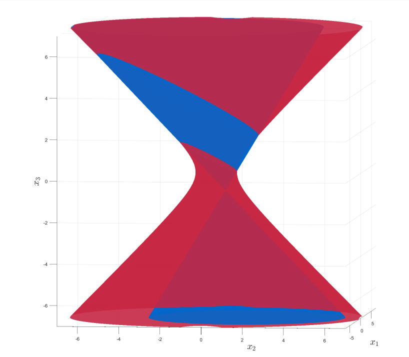


Valid conic inequality $\mathcal{Q}(\bar{\tau}_1)$, $\text{conv}(\mathcal{Q}_1 \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{Q}_1 \cap \mathcal{Q}(\tau_1)$ and $\text{conv}(\mathcal{Q}_2 \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{Q}_2 \cap \mathcal{Q}(\tau_1)$

The sets $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are bounded, $a_1 > \frac{1}{2}$



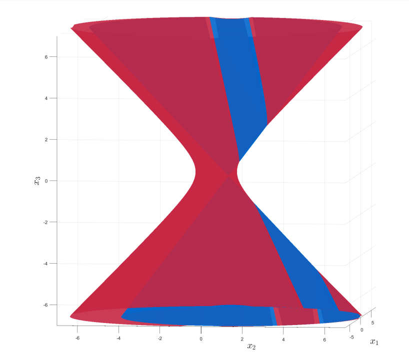
Valid conic inequality $\mathcal{Q}(\bar{\tau}_2)$



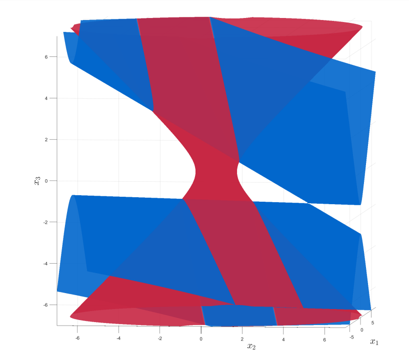
Valid conic inequality $\mathcal{Q}(\bar{\tau}_1)$

The sets $\mathcal{Q} \cap \mathcal{A}^{=}$ and $\mathcal{Q} \cap \mathcal{B}^{=}$ are unbounded

$\beta^2 \geq 1 - 2a_1^2$ and $\alpha^2 \geq 1 - 2a_1^2$



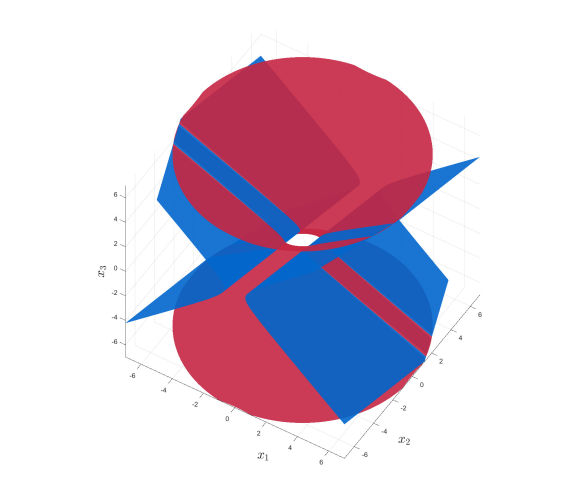
Valid conic inequality $\mathcal{Q}(\bar{\tau}_1)$, $\alpha\beta > 0$



Valid conic inequality $\mathcal{Q}(\bar{\tau}_1)$, $\alpha\beta < 0$

The sets $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are unbounded

$$\beta^2 \leq 1 - 2a_1^2 \text{ and } \alpha^2 \leq 1 - 2a_1^2$$



$\mathcal{Q}(\bar{\tau}_1)$ is a cylinder defined by a hyperboloid of one sheet

Conclusions

- We provided valid inequalities for the cross-sections of a non-convex quadratic cone.
- Showed that these valid inequalities consider the DCCs for MISOCO.
- Investigate the potential to use the family of quadrics with some other quadratic sets.

The bad

Pathological disjunction for MISOCO

Definition (Shahabsafa, G., Terlaky)
Let $\mathcal{X} \in \mathbb{R}n$ be a closed convex set, and consider the disjunction $\mathcal{A} \cup \mathcal{B}$. If $\text{conv}(\mathcal{X} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{X}$, then disjunction $\mathcal{A} \cup \mathcal{B}$ is pathological for the set $\mathcal{X}$.

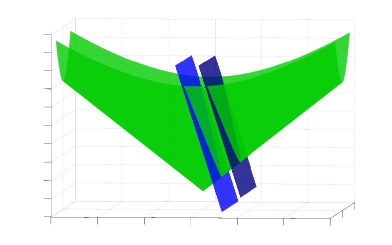
Identification of a redundant DCC for MISOCO

Corollary (Shahabsafa, G., Terlaky)

If the following two conditions are satisfied for the set $\hat{\mathcal{Q}}$ defined, and the disjunctive set, then we have a redundant DCC:

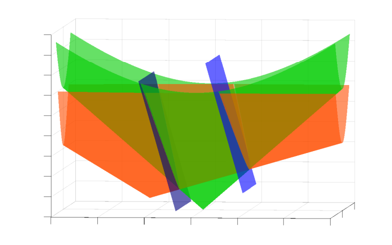
- *the matrix P has exactly $n-1$ positive eigenvalues and one negative eigenvalue, and $p^\top P^{-1} p - \rho = 0$;*
- *the vertex of the cone $v = P^{-1}p$ satisfies either $\hat{a}^\top v \geq \hat{\beta}$, or $\hat{a}^\top v \leq \hat{\alpha}$.*

Identification of a redundant DCC for MISOCO



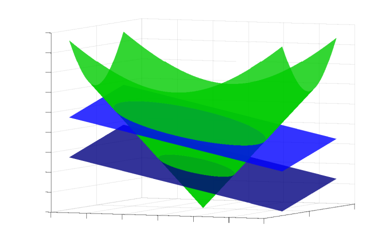
Hyperboloid intersection (Redundant DCC)

‘

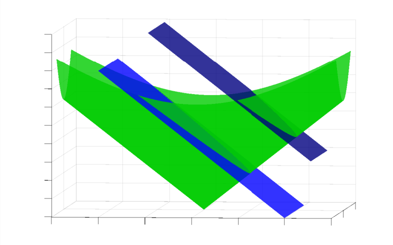


Hyperboloid intersection and the DCC (not a redundant DCC)

Identification of a redundant DCC for MISOCO



Ellipsoid intersection (Redundant DCC)



Paraboloid intersection (Redundant DCC)

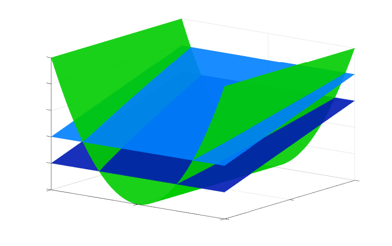
Identification of a redundant DCyC for MISOCO

Corollary (Shahabsafa, G., Terlaky)

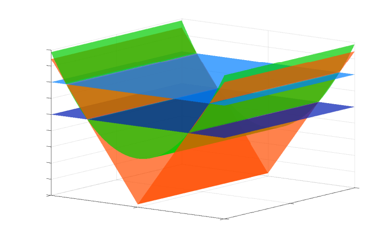
Consider the set $\hat{\mathcal{Q}}$, as defined, and a disjunction. We have a cylindrical redundant DCyC if the following two conditions are satisfied:

- *System* $\begin{bmatrix} P & p \end{bmatrix}^\top d = 0$, *for* $d \neq 0$, *has a solution.*
- *System* $\begin{bmatrix} P & p \end{bmatrix} y = \hat{a}$, *for* $y \in \mathbb{R}^{\ell+1}$, *does not have a solution.*

Identification of a redundant DCyC for MISOCO



A cylindrical redundant DCyC



Not a cylindrical redundant DCyC

Conclusions

- We presented two fundamental pathological cases, which help to identify when a DCC is redundant.
- The identification of the pathological cases is important for an efficient implementation of DCCs or derivation of them.
- The identification of the pathological cases of DCCs for MISOCO highlights both the limitations and the opportunities for the efficient implementation of the DCCs.

Implementation

Implementation challenges

- Generating DCC may messes up the structure of the problem, the matrices associated with the cuts are usually dense.
- DCC generation brings numerical challenges.
- Adding DCC may increase the solution time of the linear relaxations.
- No efficient warm start is available for interior point methods.

Constrained layout problems, Bonami et al. 2008

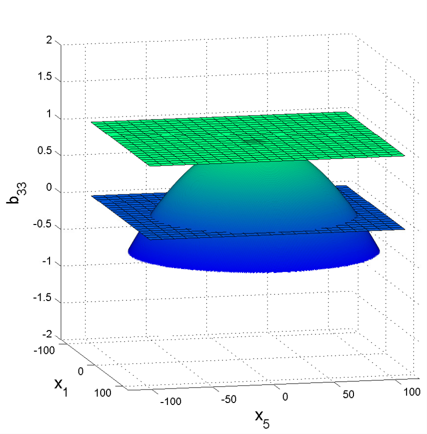
- Quadratic constraints corresponding to Euclidean-distance

$$(x_1 - 17.5)^2 + (x_5 - 7)^2 + 6814 * b_{33} \leq 6850.$$

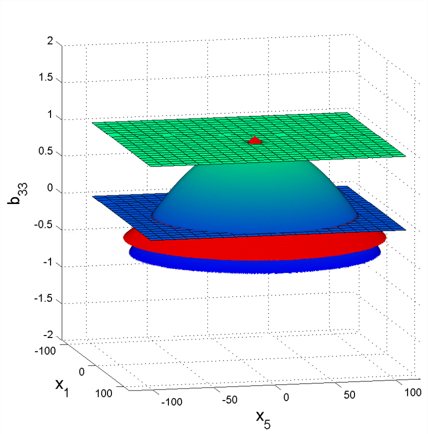
- All integer variables are binary, for example in the illustrative constraint the binary variable is b_{33}.

	0203M	0204M	0205M	0303M	0304M	0305M
Var	31	52	81	34	57	86
Binary	18	21	50	21	36	55
Constraints	55	91	136	67	107	156
Quad	24	32	40	36	48	60

Constrained layout problems, Bonami et al. 2008



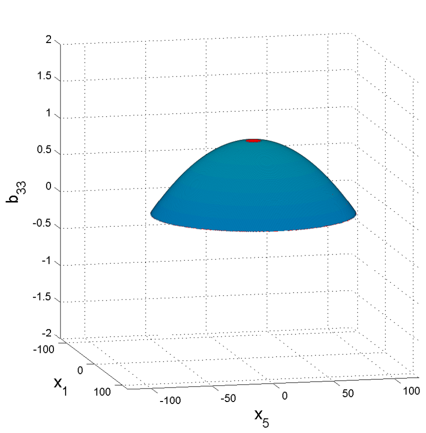
CLay Quadratic Constraints



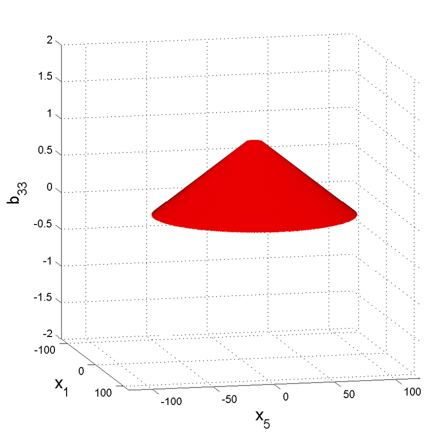
DCC cut

Constrained layout problems, Bonami et al. 2008

This could be done in the preprocessing phase



CLay Quadratic Constraints

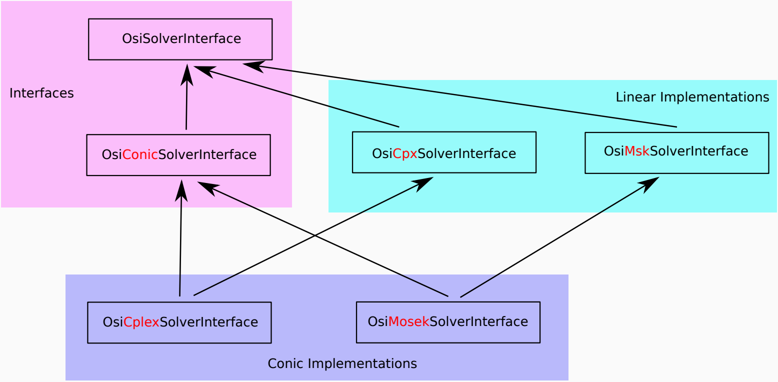


DCC cut

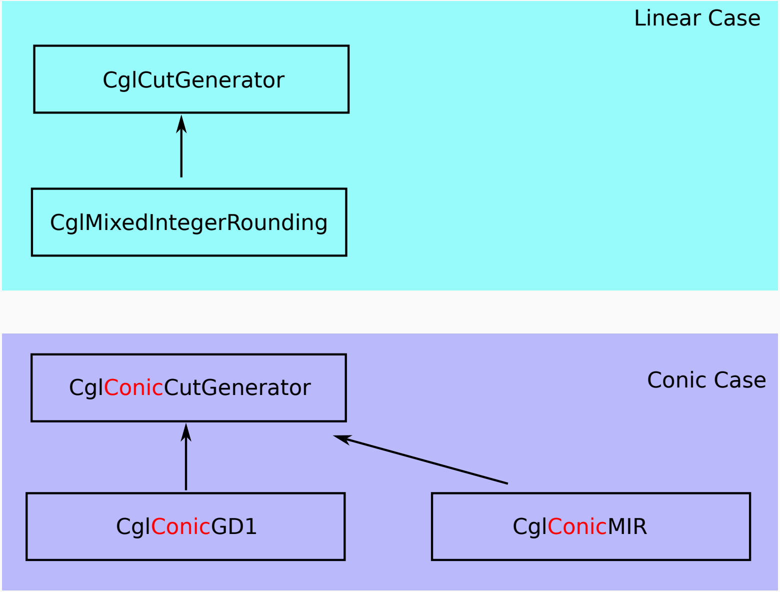
COIN-OR framework - Aykut Bulut and Ted Ralphs

- OsiConic: A generic interface class for SOCP solvers. This interface provides a way to build and solve SOCPs that is uniform across a variety of solvers, as well as a standard interface for querying the results.
- OsiXxxxx: Implementations of the interface for various open source and commercial solvers.
- COLA: A solver for SOCP that implements the cutting-plane Algorithm.
- CglConic: A library of procedures for generating valid inequalities for MISOCP.
- DisCO: A solver library for MISOCP that uses all the libraries mentioned. This library implements classical branch-and-bound type of algorithm and and outer approximation branch-and-cut algorithm.

Osi comercial



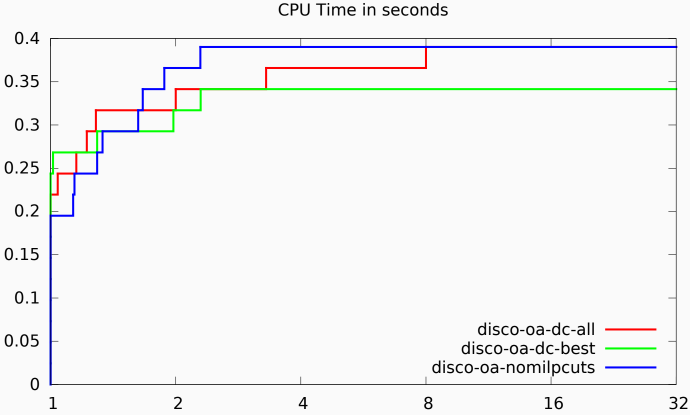
CglConic, A Cut Library for MISCOP



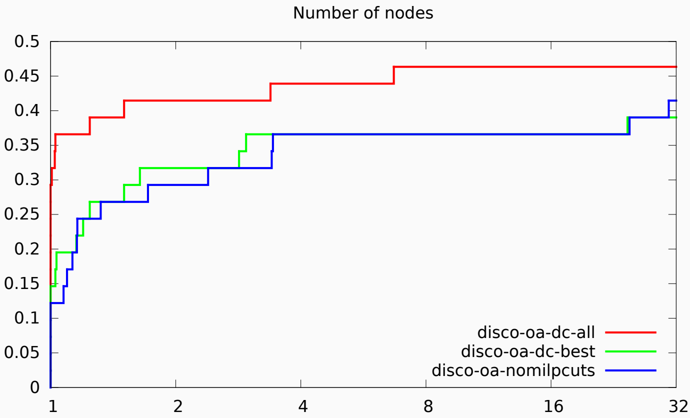
COLA statistics on Góez's random instances

instance	NC	LC	NUMLP	CPU
r12c15k5i10	5	3	5	0.01
r14c18k3i9	3	6	16	0.01
r17c30k3i12	3	10	74	0.07
r17c20k5i15	5	4	4	0.0
r22c30k10i20	10	3	8	0.02
r22c40k10i20	10	4	22	0.03
r23c45k3i21	3	15	148	0.25
r27c50k5i25	5	10	77	0.11
r32c45k15i30	15	3	6	0.0
r32c60k15i30	15	4	32	0.02
r52c75k5i35	5	15	74	0.15

Performance Profile of CPU Time using bb-lp with disjunctive cuts



Performance Profile of Number of Nodes Processed using bb-lp with disjunctive cuts



Conclusions and future work

Conclusions and future work

- We provided an extension of disjunctive programming to MISOCO problems.
- We were able to provide closed forms for the derivation of DCCs for MISOCO problems.
- This work gives a full characterization of DCCs for MISOCO problems when using parallel disjunctions.
- We provided valid inequalities for the cross-sections of a non-convex quadratic cone and a one sheet hyperboloid.
- Investigate the potential to use the family of quadrics with some other quadratic sets.
- Investigate the computational potential of this inequalities.

References

- Shahabsafa, M., Góez, J.C., Terlaky, T., On Pathological Disjunctions and Redundant Disjunctive Conic Cuts. Forthcoming.
- Aykut Bulut and Ted K. Ralphs. Cut generation library for conic problems. https://github.com/aykutbulut/CGL-CONIC. 2016
- Belotti, P., Góez, J.C., Pólik, I., Ralphs, T., Terlaky, T., A Complete Characterization of Disjunctive Conic Cuts for Mixed Integer Second Order Cone Optimization. In *Discrete optimization*, 2017.
- Belotti, P., Góez, J.C., Pólik, I., Ralphs, T., Terlaky, T., A conic representation of the convex hull of disjunctive sets and conic cuts for integer second order cone optimization. In *Numerical Analysis and Optimization, NAO-III, Muscat, Oman, January 2014*, 2015.
- Góez, J.C., Mixed integer second order cone optimization: Disjunctive conic cuts, theory and experimentation. Ph.D. thesis, Lehigh Unversity, Dept. of Industrial and Systems Engineering, July, 2013.
- Belotti, P., Góez, J.C., Pólik, I., Ralphs, T., Terlaky, T., On families of quadratic surfaces having fixed intersections with two hyperplanes. In *Discrete Applied Mathematics*, 2013.

Tusen Takk!!!