Applications and Solution Approaches for Mixed-Integer Semidefinite Programming

Tristan Gally

joint work with Marc E. Pfetsch and Stefan Ulbrich

SFB 805

Control of Uncertainty in Load-Carrying Structures in Mechanical Engineering

Mixed-Integer Semidefinite Programming

- Mixed-integer semidefinite program

MISDP

$$\begin{aligned} \sup \quad & b^T y \\ \text{s.t.} \quad & C - \sum_{i=1}^{m} A_i y_i \succeq 0, \\ & \qquad\qquad y_i \in \mathbb{Z} \quad \forall\, i \in \mathcal{I} \end{aligned}$$

for symmetric matrices A_i, C

- Linear constraints, bounds, multiple blocks possible within SDP-constraint

Mixed-Integer Semidefinite Programming

- Mixed-integer semidefinite program

MISDP

$$\begin{aligned} \sup \quad & b^T y \\ \text{s.t.} \quad & C - \sum_{i=1}^{m} A_i y_i \succeq 0, \\ & y_i \in \mathbb{Z} \quad \forall\, i \in \mathcal{I} \end{aligned}$$

for symmetric matrices A_i, C

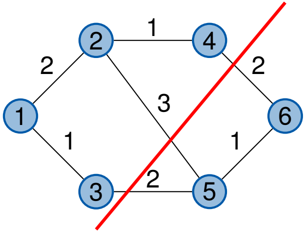
- Linear constraints, bounds, multiple blocks possible within SDP-constraint
- Efficient solvers for specific applications, but few solvers (and theory) for the general case

Contents

TECHNISCHE UNIVERSITÄT DARMSTADT

Contents

Classical Example: Max-Cut

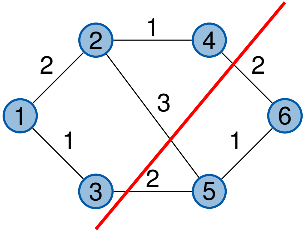


Max-Cut

Find Cut $\delta(S)$, with $S \subseteq V$ and $\{i,j\} \in \delta(S)$ iff $i \in S, j \in V \setminus S$, that maximizes

$$\sum_{\{i,j\} \in \delta(S)} c_{ij}.$$

Classical Example: Max-Cut



Max-Cut

Find Cut $\delta(S)$, with $S \subseteq V$ and $\{i,j\} \in \delta(S)$ iff $i \in S, j \in V \setminus S$, that maximizes

$$\sum_{\{i,j\} \in \delta(S)} c_{ij}.$$

Using variables $(x_i)_{i \in V} \in \{-1, 1\}^n$ with $x_i = 1 \Leftrightarrow i \in S$, this is equivalent to

Max-Cut MIQP

$$\max \quad \sum_{i<j} c_{ij} \frac{1 - x_i x_j}{2}$$

$$\text{s.t.} \quad x_i \in \{-1, 1\} \quad \forall\, i \leq n$$

Classical Example: Max-Cut

TECHNISCHE UNIVERSITÄT DARMSTADT

$$\sum_{i<j} c_{ij} \frac{1 - x_i x_j}{2} = \frac{1}{4} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} c_{ij} x_i x_i - \sum_{j=1}^{n} c_{ij} x_i x_j \right)$$

$$= \frac{1}{4} x^T (Diag(C\mathbb{1}) - C) x$$

Classical Example: Max-Cut

TECHNISCHE UNIVERSITÄT DARMSTADT

$$\sum_{i<j} c_{ij} \frac{1 - x_i x_j}{2} = \frac{1}{4} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} c_{ij} x_i x_i - \sum_{j=1}^{n} c_{ij} x_i x_j \right)$$
$$= \frac{1}{4} x^T (Diag(C\mathbb{1}) - C) x$$

With $X := xx^T$ (and notation $A \bullet B := \mathrm{Tr}(AB) = \sum_{ij} A_{ij} B_{ij}$), this is equivalent to

Max-Cut Rk1-MISDP [Poljak, Rendl 1995]

$$\begin{aligned} \max \quad & \frac{1}{4}(Diag(C\mathbb{1}) - C) \bullet X \\ \text{s.t.} \quad & \mathrm{diag}(X) = \mathbb{1} \\ & \mathrm{Rank}(X) = 1 \\ & X \succeq 0 \\ & X_{ij} \in \{-1, 1\} \end{aligned}$$

Classical Example: Max-Cut

Max-Cut Rk1-MISDP

$$
\begin{aligned}
\max \quad & \frac{1}{4}(Diag(C\mathbb{1}) - C) \bullet X \\
\text{s.t.} \quad & \operatorname{diag}(X) = \mathbb{1} \\
& \operatorname{Rank}(X) = 1 \\
& X \succeq 0 \\
& X_{ij} \in \{-1, 1\}
\end{aligned}
$$

- Relaxation still non-convex because of rank constraint

Classical Example: Max-Cut

TECHNISCHE UNIVERSITÄT DARMSTADT

Max-Cut MISDP

$$\begin{aligned} \max \quad & \frac{1}{4}(Diag(C\mathbb{1}) - C) \bullet X \\ \text{s.t.} \quad & \mathrm{diag}(X) = \mathbb{1} \\ & \cancel{\mathrm{Rank}(X) = 1} \\ & X \succeq 0 \\ & X_{ij} \in \{-1, 1\} \end{aligned}$$

- Relaxation still non-convex because of rank constraint

Theorem [Laurent, Poljak 1995]

Every integral solution satisfies Rank(X) = 1.

Compressed Sensing

- Task: find sparsest solution to underdetermined system of linear equations, i.e. a solution of

ℓ_0-Minimization

$$\begin{aligned} \min \quad & \|x\|_0 \\ \text{s.t.} \quad & Ax = b \\ & x \in \mathbb{R}^n \end{aligned}$$

where $\|x\|_0 := |supp(x)|$.

Compressed Sensing

TECHNISCHE UNIVERSITÄT DARMSTADT

- Task: find sparsest solution to underdetermined system of linear equations, i.e. a solution of

ℓ_0-Minimization

$$\begin{aligned} \min \quad & \|x\|_0 \\ \text{s.t.} \quad & Ax = b \\ & x \in \mathbb{R}^n \end{aligned}$$

where $\|x\|_0 := |supp(x)|$.

- Under certain conditions on A, this is equivalent to

ℓ_1-Minimization

$$\begin{aligned} \min \quad & \|x\|_1 \\ \text{s.t.} \quad & Ax = b \\ & x \in \mathbb{R}^n \end{aligned}$$

Compressed Sensing

One such condition is the (asymmetric) restricted isometry property (RIP):

$$\alpha_k^2 \|x\|_2^2 \leq \|Ax\|_2^2 \leq \beta_k^2 \|x\|_2^2 \qquad \forall x : \|x\|_0 \leq k$$

Compressed Sensing

One such condition is the (asymmetric) restricted isometry property (RIP):

$$\alpha_k^2 \|x\|_2^2 \leq \|Ax\|_2^2 \leq \beta_k^2 \|x\|_2^2 \qquad \forall x : \|x\|_0 \leq k$$

Theorem [Foucart, Lai 2008]

If $Ax = b$ has a solution x with $\|x\|_0 \leq k$ and the RIP of order $2k$ holds for

$$\frac{\beta_{2k}^2}{\alpha_{2k}^2} < 4\sqrt{2} - 3 \approx 2.6569,$$

then x is the unique solution of both the ℓ_0- and the ℓ_1-optimization problem.

Compressed Sensing

The optimal constant α_k^2 (and similarly β_k^2) for

$$\alpha_k^2 \|x\|_2^2 \leq \|Ax\|_2^2 \leq \beta_k^2 \|x\|_2^2 \qquad \forall x : \|x\|_0 \leq k$$

can be computed via the following non-convex rank-constrained MISDP:

RIP-Rk1-MISDP

$$\begin{aligned}
\min \quad & \mathrm{Tr}(A^T A X) \\
\text{s.t.} \quad & \mathrm{Tr}(X) = 1 \\
& -z_j \leq X_{jj} \leq z_j \quad \forall\, j \leq n \\
& \sum_{j=1}^{n} z_j \leq k \\
& \mathrm{Rank}(X) = 1 \\
& X \succeq 0 \\
& z \in \{0,1\}^n
\end{aligned}$$

Compressed Sensing

TECHNISCHE UNIVERSITÄT DARMSTADT

RIP-MISDP

$$\begin{aligned} \min \quad & \mathrm{Tr}(A^T A X) \\ \text{s.t.} \quad & \mathrm{Tr}(X) = 1 \\ & -z_j \leq X_{jj} \leq z_j \quad \forall j \leq n \\ & \sum_{j=1}^{n} z_j \leq k \\ & \xcancel{\mathrm{Rank}(X) = 1} \\ & X \succeq 0 \\ & z \in \{0,1\}^n \end{aligned}$$

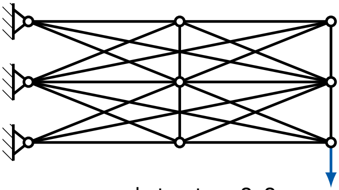
Theorem [G., Pfetsch 2016]

There always exists an optimal solution for (RIP-MISDP) with $\mathrm{Rank}(X) = 1$.

Truss Topology Design

TECHNISCHE UNIVERSITÄT DARMSTADT

- n nodes $V = \left\{ v_i \in \mathbb{R}^d : i = 1, \dots, n \right\}$
- n_f free nodes $V_f \subset V$
- m possible bars
  $E \subseteq \left\{ \left\{ v_i, v_j \right\} : i \neq j \right\}, |E| = m$
- Force $f \in \mathbb{R}^{d_f}$ for $d_f = d \cdot n_f$

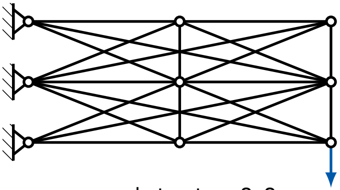


ground structure 3x3

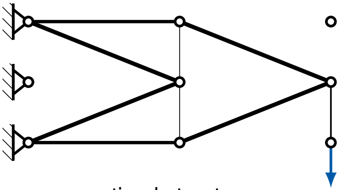
Truss Topology Design

TECHNISCHE UNIVERSITÄT DARMSTADT

- n nodes $V = \left\{ v_i \in \mathbb{R}^d : i = 1, \dots, n \right\}$
- n_f free nodes $V_f \subset V$
- m possible bars
  $E \subseteq \left\{ \left\{ v_i, v_j \right\} : i \neq j \right\}, \left| E \right| = m$
- Force $f \in \mathbb{R}^{d_f}$ for $d_f = d \cdot n_f$
- Cross-sectional areas $x \in \mathbb{R}_+^m$ for bars that minimize the volume while creating a “stable” truss
- Stability is measured by the compliance $\frac{1}{2} f^T u$ with node displacements u



ground structure 3x3



optimal structure

Truss Topology Design

TECHNISCHE UNIVERSITÄT DARMSTADT

TTD-SDP [Ben-Tal, Nemirovski 1997]

$$\min \sum_{e \in E} \ell_e x_e$$

$$\text{s.t.} \quad \begin{pmatrix} 2C_{\max} & f^T \\ f & A(x) \end{pmatrix} \succeq 0$$

$$x_e \geq 0 \quad \forall e \in E$$

- E : set of possible bars
- ℓ_e : length of bar e
- x : cross-sectional areas
- f : external force
- $C_{\max}$: upper bound on compliance
- A_e: bar stiffness matrices

with stiffness matrix $A(x) = \sum_{e \in E} A_e \ell_e x_e$.

Truss Topology Design

TECHNISCHE UNIVERSITÄT DARMSTADT

- In practice, we won't be able to produce/buy bars of any desired size.

$\Rightarrow$ Only allow cross-sectional areas from a discrete set $\mathcal{A}$.

Discrete Optimization

Truss Topology Design

TECHNISCHE UNIVERSITÄT DARMSTADT

- In practice, we won't be able to produce/buy bars of any desired size.

$\Rightarrow$ Only allow cross-sectional areas from a discrete set $\mathcal{A}$.

TTD-MISDP [Kočvara 2010, Mars 2013]

$$
\begin{aligned}
\min \quad & \sum_{e \in E} \ell_e \sum_{a \in \mathcal{A}} a x_e^a \\
\text{s.t.} \quad & \begin{pmatrix} 2C_{\max} & f^T \\ f & A(x) \end{pmatrix} \succeq 0 \\
& \sum_{a \in \mathcal{A}} x_e^a \leq 1 \qquad \forall e \in E \\
& x_e^a \in \{0, 1\} \qquad \forall e \in E, a \in \mathcal{A},
\end{aligned}
$$

where $A(x) = \sum_{e \in E} \sum_{a \in \mathcal{A}} A_e \, \ell_e \, a \, x_e^a$.

Further Applications

TECHNISCHE UNIVERSITÄT DARMSTADT

- AC power flow
  - Transmission switching problems
  - Unit commitment problems
- Cardinality-constrained least-squares
- Minimum *k*-partitioning
- Quadratic assignment problems (including TSP as special case)
- Robustification of physical parameters in gas networks
- Subset selection for eliminating multicollinearity
- ...

Contents

Outer Approximation / Cutting Planes

TECHNISCHE UNIVERSITÄT DARMSTADT

- Idea: Solve LP/MIP and enforce SDP-constraint via linear cuts

Discrete Optimization

Outer Approximation / Cutting Planes

TECHNISCHE UNIVERSITÄT DARMSTADT

- Idea: Solve LP/MIP and enforce SDP-constraint via linear cuts
- Cutting plane approach (Kelley 1960):
  - Solve a single MIP
  - In each node add cuts to enforce nonlinear constraints and resolve LP

Outer Approximation / Cutting Planes

TECHNISCHE UNIVERSITÄT DARMSTADT

- Idea: Solve LP/MIP and enforce SDP-constraint via linear cuts
- Cutting plane approach (Kelley 1960):
  - Solve a single MIP
  - In each node add cuts to enforce nonlinear constraints and resolve LP
- Outer Approximation (Quesada/Grossmann 1992):
  - Solve MIP (without nonlinear constraints) to optimality
  - Solve continuous relaxation for fixed integer variables
  - If objectives do not agree, update polyhedral approximation
  - Resolve MIP and continue iterating

Enforcing the SDP-Constraint

- For convex MINLP one usually uses gradient cuts

$$g_j(\overline{x}) + \nabla g_j(\overline{x})^\top (x - \overline{x}) \leq 0.$$

- But function of smallest eigenvalue is not differentiable everywhere.

Enforcing the SDP-Constraint

TECHNISCHE UNIVERSITÄT DARMSTADT

- For convex MINLP one usually uses gradient cuts

$$g_j(\overline{x}) + \nabla g_j(\overline{x})^\top (x - \overline{x}) \leq 0.$$

- But function of smallest eigenvalue is not differentiable everywhere.

$\Rightarrow$ Instead use characterization $X \succeq 0 \quad \Leftrightarrow \quad u^\top X u \geq 0$ for all $u \in \mathbb{R}^n$

- If $Z := C - \sum_{i=1}^{m} A_i y_i^* \not\succeq 0$, compute eigenvector v to smallest eigenvalue. Then

$$v^\top Z v \geq 0$$

is a valid linear cut that cuts off y^*.

Discrete Optimization

Cutting Plane Approach: MISOCP vs. MISDP

- Successfully used by many commercial solvers for mixed-integer second-order cone
- Outer approximation for SOCPs possible with polynomial number of cuts (Ben-Tal/Nemirovski 2001)
- Outer approximation for SDPs needs exponential number of cuts (Braun et al. 2015)

SDP-based Branch-and-Bound

TECHNISCHE UNIVERSITÄT DARMSTADT

- Relax integrality instead of SDP-constraint
- Need to solve a continuous SDP in each branch-and-bound node
- Relaxations can be solved by problem-specific approaches (e.g. conic bundle or low-rank methods) or interior-point
- Need to satisfy convergence assumptions of SDP-solvers

Strong Duality in SDP

TECHNISCHE UNIVERSITÄT DARMSTADT

Dual SDP (D)

$$\begin{aligned} \sup \quad & b^T y \\ \text{s.t.} \quad & C - \sum_{i=1}^{m} A_i y_i \succeq 0 \\ & y \in \mathbb{R}^m \end{aligned}$$

Primal SDP (P)

$$\begin{aligned} \inf \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X = b_i \quad \forall\, i \leq m \\ & X \succeq 0 \end{aligned}$$

Strong Duality in SDP

TECHNISCHE UNIVERSITÄT DARMSTADT

Dual SDP (D)

$$\begin{aligned} \sup \quad & b^T y \\ \text{s.t.} \quad & C - \sum_{i=1}^{m} A_i y_i \succeq 0 \\ & y \in \mathbb{R}^m \end{aligned}$$

Primal SDP (P)

$$\begin{aligned} \inf \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X = b_i \quad \forall\, i \leq m \\ & X \succeq 0 \end{aligned}$$

- Strong Duality holds if Slater condition holds for (P) or (D), i.e., there exists a feasible $X \succ 0$ for (P) or y such that $C - \sum_{i=1}^{m} A_i y_i \succ 0$ in (D).

Strong Duality in SDP

TECHNISCHE UNIVERSITÄT DARMSTADT

Dual SDP (D)

$$\begin{aligned} \sup \quad & b^T y \\ \text{s.t.} \quad & C - \sum_{i=1}^{m} A_i y_i \succeq 0 \\ & y \in \mathbb{R}^m \end{aligned}$$

Primal SDP (P)

$$\begin{aligned} \inf \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X = b_i \quad \forall\, i \leq m \\ & X \succeq 0 \end{aligned}$$

- Strong Duality holds if Slater condition holds for (P) or (D), i.e., there exists a feasible $X \succ 0$ for (P) or y such that $C - \sum_{i=1}^{m} A_i y_i \succ 0$ in (D).
- If Slater holds for (P), optimal objective of (D) is attained and vice versa.

Strong Duality in SDP

Dual SDP (D)

$$\begin{aligned} \sup \quad & b^T y \\ \text{s.t.} \quad & C - \sum_{i=1}^{m} A_i y_i \succeq 0 \\ & y \in \mathbb{R}^m \end{aligned}$$

Primal SDP (P)

$$\begin{aligned} \inf \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X = b_i \quad \forall\, i \leq m \\ & X \succeq 0 \end{aligned}$$

- Strong Duality holds if Slater condition holds for (P) or (D), i.e., there exists a feasible $X \succ 0$ for (P) or y such that $C - \sum_{i=1}^{m} A_i y_i \succ 0$ in (D).
- If Slater holds for (P), optimal objective of (D) is attained and vice versa.
- Existence of a KKT-point is guaranteed if Slater holds for both, usual assumption of interior-point algorithms for SDP.

Strong Duality in SDP

TECHNISCHE UNIVERSITÄT DARMSTADT

Dual SDP (D)

$$\begin{aligned} \sup \quad & b^T y \\ \text{s.t.} \quad & C - \sum_{i=1}^{m} A_i y_i \succeq 0 \\ & y \in \mathbb{R}^m \end{aligned}$$

Primal SDP (P)

$$\begin{aligned} \inf \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X = b_i \quad \forall\, i \leq m \\ & X \succeq 0 \end{aligned}$$

- Strong Duality holds if Slater condition holds for (P) or (D), i.e., there exists a feasible $X \succ 0$ for (P) or y such that $C - \sum_{i=1}^{m} A_i y_i \succ 0$ in (D).
- If Slater holds for (P), optimal objective of (D) is attained and vice versa.
- Existence of a KKT-point is guaranteed if Slater holds for both, usual assumption of interior-point algorithms for SDP.
- Need to assume this for root node. But is this enough or can these assumptions be lost through branching?

Strong Duality in Branch-and-Bound

TECHNISCHE UNIVERSITÄT DARMSTADT

Theorem [G., Pfetsch, Ulbrich 2016]

Let (D_+) be the problem formed by adding a linear constraint to (D). If

- strong duality holds for (P) and (D),
- the set of optimal $Z := C - \sum_{i=1}^{m} A_i y_i$ in (D) is compact and nonempty,
- the problem (D_+) is feasible,

then strong duality also holds for (D_+) and (P_+) and the set of optimal Z for (D_+) is compact and nonempty.

Strong Duality in Branch-and-Bound

TECHNISCHE UNIVERSITÄT DARMSTADT

Theorem [G., Pfetsch, Ulbrich 2016]

Let (D_+) be the problem formed by adding a linear constraint to (D). If

- strong duality holds for (P) and (D),
- the set of optimal $Z := C - \sum_{i=1}^{m} A_i y_i$ in (D) is compact and nonempty,
- the problem (D_+) is feasible,

then strong duality also holds for (D_+) and (P_+) and the set of optimal Z for (D_+) is compact and nonempty.

- Compactness of set of optimal Z also necessary for strong duality (Friberg 2016)

Strong Duality in Branch-and-Bound

TECHNISCHE UNIVERSITÄT DARMSTADT

Theorem [G., Pfetsch, Ulbrich 2016]

Let (D_+) be the problem formed by adding a linear constraint to (D). If

- strong duality holds for (P) and (D),
- the set of optimal $Z := C - \sum_{i=1}^{m} A_i y_i$ in (D) is compact and nonempty,
- the problem (D_+) is feasible,

then strong duality also holds for (D_+) and (P_+) and the set of optimal Z for (D_+) is compact and nonempty.

- Compactness of set of optimal Z also necessary for strong duality (Friberg 2016)
- Equivalent result for adding linear constraints to (P) with set of optimal X compact and nonempty and (P_+) feasible

Slater Condition in Branch-and-Bound

TECHNISCHE UNIVERSITÄT DARMSTADT

Proposition [G., Pfetsch, Ulbrich 2016]

After adding a linear constraint $\sum_{i=1}^{m} a_i y_i \geq c$ (or $\leq$ or $=$) to (D), if (P) satisfies the Slater condition and the coefficient vector a satisfies $a \in \text{Range}(\mathcal{A})$, for $\mathcal{A} : S_n \to \mathbb{R}^m$, $X \mapsto (A_i \bullet X)_{i \in [m]}$, then the Slater condition also holds for (P_+).

Slater Condition in Branch-and-Bound

TECHNISCHE UNIVERSITÄT DARMSTADT

Proposition [G., Pfetsch, Ulbrich 2016]

After adding a linear constraint $\sum_{i=1}^{m} a_i y_i \geq c$ (or $\leq$ or $=$) to (D), if (P) satisfies the Slater condition and the coefficient vector a satisfies $a \in \text{Range}(\mathcal{A})$, for $\mathcal{A} : S_n \to \mathbb{R}^m$, $X \mapsto (A_i \bullet X)_{i \in [m]}$, then the Slater condition also holds for (P_+).

- $a \in \text{Range}(\mathcal{A})$ is implied by linear independence of A_i.

Slater Condition in Branch-and-Bound

TECHNISCHE UNIVERSITÄT DARMSTADT

Proposition [G., Pfetsch, Ulbrich 2016]

After adding a linear constraint $\sum_{i=1}^{m} a_i y_i \geq c$ (or $\leq$ or $=$) to (D), if (P) satisfies the Slater condition and the coefficient vector a satisfies $a \in \text{Range}(\mathcal{A})$, for $\mathcal{A} : S_n \to \mathbb{R}^m$, $X \mapsto (A_i \bullet X)_{i \in [m]}$, then the Slater condition also holds for (P_+).

- $a \in \text{Range}(\mathcal{A})$ is implied by linear independence of A_i.
- Dual Slater condition is preserved after adding linear constraint to (P) (without additional assumptions on the coefficients).

KKT-condition in Branch-and-Bound

KKT-points may get lost after branching:

(D)

$$\begin{aligned} \sup \quad & 2\,y_1 - y_2 \\ \text{s.t.} \quad & \begin{pmatrix} 0.5 & -y_1 \\ -y_1 & y_2 \end{pmatrix} \succeq 0, \end{aligned}$$

(P)

$$\begin{aligned} \inf \quad & 0.5\,X_{11} \\ \text{s.t.} \quad & \begin{pmatrix} X_{11} & 1 \\ 1 & 1 \end{pmatrix} \succeq 0, \end{aligned}$$

- Strictly feasible solutions given by $y = (0, 0.5)$, $X_{11} = 2$
- Optimal objective of 0.5 attained (only) for $y = (0.5, 0.5)$, $X_{11} = 1$

KKT-condition in Branch-and-Bound

TECHNISCHE
UNIVERSITÄT
DARMSTADT

After branching on y_2 and adding cut $y_2 \leq 0$:

(D_+)

$$\begin{array}{ll} \sup & 2\,y_1 - y_2 \\ \text{s.t.} & \begin{pmatrix} 0.5 & -y_1 & 0 \\ -y_1 & y_2 & 0 \\ 0 & 0 & -y_2 \end{pmatrix} \succeq 0, \end{array}$$

(P_+)

$$\begin{array}{ll} \inf & 0.5\,X_{11} \\ \text{s.t.} & \begin{pmatrix} X_{11} & 1 & X_{13} \\ 1 & X_{22} & X_{23} \\ X_{13} & X_{23} & X_{22} - 1 \end{pmatrix} \succeq 0, \end{array}$$

- Optimal objective 0 attained for $y = (0, 0)$
- Relative interior of (D_+) is empty
- (P_+) still has strictly feasible solution $X_{11} = X_{22} = 2$, $X_{13} = X_{23} = 0$
- (P_+) has minimizing sequence $X_{11} = 1/k$, $X_{22} = k$, $X_{13} = X_{23} = 0$
- No longer satisfies assumptions for convergence of interior-point solvers

Slater Condition in Practice

	Dual Slater				Primal Slater		
Problem	✓	✗	inf	?	✓	✗	?
CLS	55.23 %	3.26 %	41.46 %	0.04 %	99.26 %	0.00 %	0.73 %
Mk-P	3.66 %	65.49 %	30.85 %	0.00 %	99.99 %	0.00 %	0.01 %
TTD	81.99 %	5.96 %	12.02 %	0.03 %	99.37 %	0.00 %	0.63 %
Overall	45.16 %	26.23 %	28.58 %	0.02 %	99.55 %	0.00 %	0.44 %

run on cluster of 40 Intel Xeon E5-1620 3.5 GHz processors with 4 cores and 32GB memory; using SCIP-SDP 3.0.0 and DSDP 5.8; on testset of 194 CBLIB instances

Checking Infeasibility

TECHNISCHE UNIVERSITÄT DARMSTADT

- If interior-point solver did not converge for original formulation, solve

Feasibility Check [Mars 2013]

$$\begin{aligned} \inf \quad & r \\ \text{s.t.} \quad & C - \sum_{i=1}^{m} A_i y_i + I\, r \succeq 0. \end{aligned}$$

If optimum $r^* > 0$, original problem is infeasible and node can be cut off.

Handling Failure of the Dual Slater Condition

TECHNISCHE UNIVERSITÄT DARMSTADT

- If problem is not infeasible, solve

Penalty Formulation [Benson, Ye 2008]

$$\begin{aligned} \sup \quad & b^\top y - \Gamma r \\ \text{s.t.} \quad & C - \sum_{i=1}^{m} A_i y_i + I r \succeq 0, \\ & r \geq 0 \end{aligned}$$

for sufficiently large Γ to compute an upper bound.

- If optimal $r^* = 0$, then solution is also optimal for original problem.
- Adds constraint $\mathrm{Tr}(X) \leq \Gamma$ to primal problem, for large enough Γ also preserves primal Slater condition.

SDP-Solvers depending on Slater Condition

TECHNISCHE UNIVERSITÄT DARMSTADT

Behavior if Slater condition holds for (P) and (D)

solver	default	penalty	bound	unsucc
SDPA	90.78 %	5.50 %	0.00 %	3.73 %
DSDP	99.68 %	0.32 %	0.00 %	0.00 %
MOSEK	99.51 %	0.49 %	0.00 %	0.00 %

Behavior if Slater condition fails for (P) or (D)

solver	default	penalty	bound	unsucc
SDPA	56.15 %	1.14 %	13.00 %	29.71 %
DSDP	99.81 %	0.13 %	0.00 %	0.05 %
MOSEK	99.20 %	0.79 %	0.01 %	0.00 %

Behavior if problem is infeasible

solver	default	feas check	bound	unsucc
SDPA	46.99 %	39.46 %	4.88 %	8.67 %
DSDP	92.44 %	2.23 %	1.39 %	3.94 %
MOSEK	88.42 %	10.36 %	1.22 %	0.00 %

Contents

TECHNISCHE UNIVERSITÄT DARMSTADT

Discrete Optimization

Warmstarts

- MIP: Large savings by starting dual simplex from optimal basis of parent node.
- Interior-point solvers: Need $X \succ 0$ and $Z := C - \sum_{i=1}^{m} A_i y_i \succ 0$ for initial point.
- Not satisfied by optimal solution of parent node, which will be on boundary.
- Infeasible interior-point methods update Z and y separately, so Z doesn't necessarily need to be updated after branching, but has to be positive definite.

$\Rightarrow$ Cannot easily warmstart with unadjusted solution of parent node.

Warmstarting Techniques

TECHNISCHE UNIVERSITÄT DARMSTADT

- Starting from Earlier Iterates
- Convex Combination with Strictly Feasible Solution
- Projection onto Positive Definite Cone
- Rounding Problems

Starting from Earlier Iterates

TECHNISCHE UNIVERSITÄT DARMSTADT

- Proposed by Gondzio for MIP.
- Store earlier iterate further away from optimum but still sufficiently interior.
- First solve relaxation to sufficiently large gap ε_1 (e.g., 10^{-2}), then save current iterate and continue solving until original tolerance ε_2 (e.g., 10^{-5}) is reached.

Convex Combination with Strictly Feasible Solution

- First proposed by Helmberg and Rendl, recently revisited by Skajaa, Andersen and Ye for MIP.
- Take convex combination between optimal solution (X^*, y^*, Z^*) and strictly feasible (X^0, y^0, Z^0).
- Choose (X^0, y^0, Z^0) as default initial point like $(I, 0, I)$, possibly scaled either by maximum entry of primal/dual matrix or maximum of both.
- Also possible to compute analytic center of feasible region once in root node and use this as strictly feasible solution.

Projection onto Positive Definite Cone

TECHNISCHE UNIVERSITÄT DARMSTADT

- Project optimal solution of parent node onto set of positive definite matrices with $\lambda_{\text{min}} \geq \underline{\lambda} > 0$.

- For given optimal solution X^* (equivalently Z^*) of parent node let $V\text{Diag}(\lambda)V^\top = X^*$ be an eigenvector decomposition. Then compute

$$V\text{Diag}((\max\{\lambda_i, \underline{\lambda}\})_{i \leq n})V^\top.$$

Discrete Optimization

Rounding Problems

TECHNISCHE UNIVERSITÄT DARMSTADT

- Proposed by Çay, Pólik and Terlaky for MISOCP based on Jordan Frames
- Fix EV decomposition $V\text{Diag}(\lambda^*)V^\top = X^*$ and optimize over eigenvalues

Discrete Optimization

Rounding Problems

TECHNISCHE UNIVERSITÄT DARMSTADT

- Proposed by Çay, Pólik and Terlaky for MISOCP based on Jordan Frames
- Fix EV decomposition $V\mathrm{Diag}(\lambda^*)V^\top = X^*$ and optimize over eigenvalues
- First solve the **linear** primal rounding problem

Primal SDP (P)

$$\begin{aligned} \inf \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X = b_i \quad \forall\, i \leq m \\ & X \succeq 0 \end{aligned}$$

Primal Rounding Problem (P-R)

$$\begin{aligned} \inf \quad & C \bullet \left(V\mathrm{Diag}(\lambda)V^\top\right) \\ \text{s.t.} \quad & A_i \bullet \left(V\mathrm{Diag}(\lambda)V^\top\right) = b_i \quad \forall\, i \leq m \\ & \lambda_i \geq 0 \quad \forall\, i \leq n \end{aligned}$$

Rounding Problems

- Proposed by Çay, Pólik and Terlaky for MISOCP based on Jordan Frames
- Fix EV decomposition $V\mathrm{Diag}(\lambda^*)V^\top = X^*$ and optimize over eigenvalues
- First solve the **linear** primal rounding problem

Primal SDP (P)

$$\begin{array}{lll} \inf & C \bullet X & \\ \text{s.t.} & A_i \bullet X = b_i & \forall\, i \leq m \\ & X \succeq 0 & \end{array}$$

Primal Rounding Problem (P-R)

$$\begin{array}{lll} \inf & C \bullet \left(V\mathrm{Diag}(\lambda)V^\top\right) & \\ \text{s.t.} & A_i \bullet \left(V\mathrm{Diag}(\lambda)V^\top\right) = b_i & \forall\, i \leq m \\ & \lambda_i \geq 0 & \forall\, i \leq n \end{array}$$

- (P-R) is restriction of (P) to matrices with same eigenvectors as X^*

$$\Rightarrow \text{optval(P-R)} \geq \text{optval(P)} \geq \text{optval(D)}$$

- (P-R) unbounded $\Rightarrow$ (D) infeasible
- optval(P-R) $\leq$ cutoff bound $\Rightarrow$ (D) not optimal

Rounding Problems

TECHNISCHE UNIVERSITÄT DARMSTADT

- If (D) is not cut off, let $W\mathrm{Diag}(\mu^*)W^\top = Z^*$ be an eigenvector decomposition of the parent node and solve the corresponding linear dual rounding problem

Dual SDP (D)

$$\begin{aligned} \sup \quad & b^T y \\ \text{s.t.} \quad & C - \sum_{i=1}^m A_i y_i = Z \\ & Z \succeq 0, \quad y \in \mathbb{R}^m \end{aligned}$$

Dual Rounding Problem (D-R)

$$\begin{aligned} \sup \quad & b^T y \\ \text{s.t.} \quad & W\mathrm{Diag}(\mu)W^\top + \sum_{i=1}^m A_i y_i = C \\ & \mu_i \geq 0 \quad \forall\, i \leq n, \quad y \in \mathbb{R}^m \end{aligned}$$

Discrete Optimization

Rounding Problems

TECHNISCHE UNIVERSITÄT DARMSTADT

- If (D) is not cut off, let $W\mathrm{Diag}(\mu^*)W^\top = Z^*$ be an eigenvector decomposition of the parent node and solve the corresponding linear dual rounding problem

Dual SDP (D)

$$\begin{aligned} \sup \quad & b^T y \\ \text{s.t.} \quad & C - \sum_{i=1}^{m} A_i y_i = Z \\ & Z \succeq 0, \quad y \in \mathbb{R}^m \end{aligned}$$

Dual Rounding Problem (D-R)

$$\begin{aligned} \sup \quad & b^T y \\ \text{s.t.} \quad & W\mathrm{Diag}(\mu)W^\top + \sum_{i=1}^{m} A_i y_i = C \\ & \mu_i \geq 0 \quad \forall\, i \leq n, \quad y \in \mathbb{R}^m \end{aligned}$$

- Since (D-R) is restriction of (D) to matrices with same eigenvectors as Z^*,

$$\text{optval(D-R)} \leq \text{optval(D)} \leq \text{optval(P)} \leq \text{optval(P-R)}.$$

- optval(D-R) = optval(P-R) $\Rightarrow$ problem solved to optimality

Rounding Problems

TECHNISCHE UNIVERSITÄT DARMSTADT

- If (D) is not cut off, let $W\mathrm{Diag}(\mu^*)W^\top = Z^*$ be an eigenvector decomposition of the parent node and solve the corresponding linear dual rounding problem

Dual SDP (D)

$$\begin{aligned} \sup \quad & b^T y \\ \text{s.t.} \quad & C - \sum_{i=1}^m A_i y_i = Z \\ & Z \succeq 0, \quad y \in \mathbb{R}^m \end{aligned}$$

Dual Rounding Problem (D-R)

$$\begin{aligned} \sup \quad & b^T y \\ \text{s.t.} \quad & W\mathrm{Diag}(\mu)W^\top + \sum_{i=1}^m A_i y_i = C \\ & \mu_i \geq 0 \quad \forall\, i \leq n, \quad y \in \mathbb{R}^m \end{aligned}$$

- Since (D-R) is restriction of (D) to matrices with same eigenvectors as Z^*,

$$\mathrm{optval(D\text{-}R)} \leq \mathrm{optval(D)} \leq \mathrm{optval(P)} \leq \mathrm{optval(P\text{-}R)}.$$

- optval(D-R) = optval(P-R) $\quad\Rightarrow\quad$ problem solved to optimality
- Otherwise use convex combination to compute strictly feasible initial point.

Rounding Problems

TECHNISCHE UNIVERSITÄT DARMSTADT

testset	time	roundtime		statistics for feasible roundingproblems				infeasibility	
			opt	cutoff	warmstart	pfail	dfail	detected	undetected
CLS	229.38	101.19	0.03	0.68	0.03	0.00	1847.37	310.27	841.17
MkP	271.18	6.97	0.00	0.40	0.88	0.12	188.18	1.49	459.83
TT	102.73	17.80	0.02	44.65	284.81	0.00	13,616.42	24.21	1805.33
CS	166.69	86.72	0.17	6022.54	4794.20	0.00	0.02	0.01	0.37

run on cluster of 40 Intel Xeon E5-1620 3.5 GHz processors with 4 cores and 32GB memory; time limit of 3600 seconds; times as shifted geometric means, SDPs solved using SDPA 7.4.0; $\gamma = 0.5$

Comparison of Warmstarting Techniques

TECHNISCHE UNIVERSITÄT DARMSTADT

settings	solved	time	sdpiter
no warmstart	290	117.85	22,827.93
unadjusted warmstart	126	821.82	–
earlier iterate: gap 0.01	172	396.93	–
earlier iterate: gap 0.5	252	213.88	26,923.91
convcomb: 0.01 scaled (pdsame) id	288	113.60	19,697.25
convcomb: 0.5 scaled (pddiff) id	289	108.60	18,307.29
convcomb: 0.5 scaled (pdsame) id	290	109.92	19,684.70
convcomb: 0.5 analcent	288	140.21	25,351.48
projection	289	112.87	20,195.03
roundingprob 0.5 id	281	180.95	16,955.37
roundingprob inf only	289	159.66	18,521.50

run on cluster of 40 Intel Xeon E5-1620 3.5 GHz processors with 4 cores and 32GB memory; time limit of 3600 seconds; times (and iterations) as shifted geometric means (over instances solved by all settings except unadjusted warmstart and preoptimal), SDPs solved using SDPA 7.4.0

Comparison of Warmstarting Techniques

TECHNISCHE UNIVERSITÄT DARMSTADT

Speedup for conv 0.01 pdsame

testset	solved	time	sdpiter
CLS	0	-11.4 %	-19.3 %
MkP	+1	-17.2 %	-21.3 %
TT	-3	+17.5 %	+34.0 %
CS	0	-9.4 %	-18.3 %

Speedup for conv 0.5 pddiff

testset	solved	time	sdpiter
CLS	0	-6.7 %	-12.2 %
MkP	+1	-0.1 %	-10.2 %
TT	-2	+33.5 %	+2.8 %
CS	0	-27.2 %	-30.5 %

Speedup for conv 0.5 pdsame

testset	solved	time	sdpiter
CLS	-1	-9.9 %	-19.7 %
MkP	+2	-8.6 %	+0.5 %
TT	-1	+15.4 %	-5.3 %
CS	0	-13.3 %	-13.8 %

Speedup for projection

testset	solved	time	sdpiter
CLS	-1	-1.7 %	-6.4 %
MkP	+1	+5.7 %	+12.2 %
TT	-1	+7.9 %	-2.7 %
CS	0	-15.8 %	-22.1 %

run on cluster of 40 Intel Xeon E5-1620 3.5 GHz processors with 4 cores and 32GB memory; time limit of 3600 seconds; times (and iterations) as shifted geometric means (over instances solved by all settings except unadjusted warmstart and preoptimal), SDPs solved using SDPA 7.4.0

Contents

TECHNISCHE UNIVERSITÄT DARMSTADT

Dual Fixing

- Generalization of reduced-cost fixing for MILPs
- Used for interior-point LP-solvers by Mitchell (1997), primal MISDPs by Helmberg (2000) and general MINLPs by Vigerske (2012)

Dual Fixing

- Generalization of reduced-cost fixing for MILPs
- Used for interior-point LP-solvers by Mitchell (1997), primal MISDPs by Helmberg (2000) and general MINLPs by Vigerske (2012)

Theorem [G., Pfetsch, Ulbrich 2016]

- (X, W, V): Primal feasible solution, where W, V are primal variables corresponding to variable bounds ℓ, u in the dual
- f: Corresponding primal objective value
- L: Lower bound on the optimal objective value of the MISDP

Then for every optimal solution of the MISDP

$$y_j \leq \ell_j + \frac{f - L}{W_{jj}} \quad \text{if } \ell_j > -\infty \qquad \text{and} \qquad y_j \geq u_j - \frac{f - L}{V_{jj}} \quad \text{if } u_j < \infty$$

- If $f - L < W_{jj}$ for binary y_j, it can be fixed to 0, if $f - L < V_{jj}$, then $y_j = 1$.

Dual Fixing

TECHNISCHE UNIVERSITÄT DARMSTADT

- Generalization of reduced-cost fixing for MILPs
- Used for interior-point LP-solvers by Mitchell (1997), primal MISDPs by Helmberg (2000) and general MINLPs by Vigerske (2012)

Theorem [G., Pfetsch, Ulbrich 2016]

- (X, W, V): Primal feasible solution, where W, V are primal variables corresponding to variable bounds ℓ, u in the dual
- f: Corresponding primal objective value
- L: Lower bound on the optimal objective value of the MISDP

Then for every optimal solution of the MISDP

$$y_j \leq \ell_j + \frac{f - L}{W_{jj}} \quad \text{if } \ell_j > -\infty \qquad \text{and} \qquad y_j \geq u_j - \frac{f - L}{V_{jj}} \quad \text{if } u_j < \infty$$

- If $f - L < W_{jj}$ for binary y_j, it can be fixed to 0, if $f - L < V_{jj}$, then $y_j = 1$.
- 6% reduction of B&B-nodes, 26% speedup

Contents

MISDP-Solvers

Nonlinear branch-and-bound

- SCIP-SDP 3.1 (nonlinear branch-and-bound)
  - Our implementation, using SCIP as B&B-framework
- YALMIP-BNB R20170921
  - MATLAB toolbox for rapid prototyping

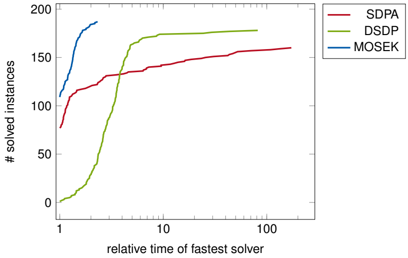
Cutting plane / outer approximation approaches

- SCIP-SDP 3.1 (LP-based cutting planes)
- YALMIP-CUTSDP R20170921
- Pajarito 0.5
  - Julia implementation for mixed-integer convex including MISDP
  - MIP-solver-drives version (single MIP with SDP solves for stronger cuts)

Which SDP-Solver to use for the Relaxations?

TECHNISCHE UNIVERSITÄT DARMSTADT

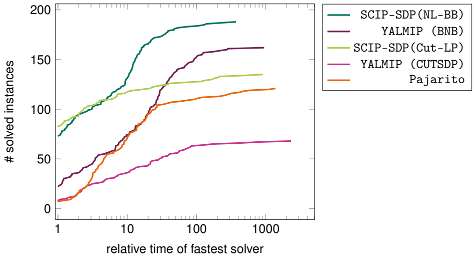
solver	solved	time
SDPA	161	136.3
DSDP	175	157.3
MOSEK	187	64.9



run on cluster of 40 Intel Xeon E5-1620 3.5 GHz processors with 4 cores and 32GB memory; time limit of 3600 seconds, times as shifted geometric means

Discrete Optimization

Comparison of MISDP-Solvers



solver	CLS		Mk-P		TTD		Total	
	solved	time	solved	time	solved	time	solved	time
SCIP-SDP(NL-BB)	63	104.3	68	38.3	57	65.2	188	63.9
YALMIP(BNB)	62	195.9	64	61.7	36	537.5	162	181.5
SCIP-SDP(Cut-LP)	65	8.3	31	614.8	39	230.5	135	132.6
YALMIP(CUTSDP)	31	525.1	15	1145.7	22	945.9	68	832.0
Pajarito	65	64.2	13	1577.5	43	220.7	121	303.3

run on 8-core Intel i7-4770 CPU with 3.4 GHz and 16GB memory; time limit of 3600 seconds, times as shifted geometric means, SDPs solved using MOSEK 8.1.0.25, MIPs/LPs using CPLEX 12.6.1

Contents

TECHNISCHE UNIVERSITÄT DARMSTADT

Conclusion & Outlook

Conclusion

- MISDPs can be solved by generic framework
- Primal Slater condition inherited in MISDP, Dual Slater may get lost
- Warmstarting is possible and can help for some applications

Conclusion & Outlook

Conclusion

- MISDPs can be solved by generic framework
- Primal Slater condition inherited in MISDP, Dual Slater may get lost
- Warmstarting is possible and can help for some applications

Outlook

- Cutting Planes
  - Chvátal-Gomory / knapsack cuts portable to MISDP, but generation less clear

Conclusion & Outlook

Conclusion

- MISDPs can be solved by generic framework
- Primal Slater condition inherited in MISDP, Dual Slater may get lost
- Warmstarting is possible and can help for some applications

Outlook

- Cutting Planes
  - Chvátal-Gomory / knapsack cuts portable to MISDP, but generation less clear
- Facial Reduction
  - Project on minimal face of psd-cone if Slater condition fails
  - Projection as solution of homogeneous self-dual model
  - Optimization over face of psd-cone can again be formulated as SDP

SCIP-SDP is available in source code at

`http://www.opt.tu-darmstadt.de/scipsdp/`

Thank you for your attention!

References I

Aharon Ben-Tal and Arkadii Nemirovski.
Robust truss topology design via semidefinite programming.
SIAM Journal on Optimization, 7(4):991–1016, 1997.

Aharon Ben-Tal and Arkadii Nemirovski.
On polyhedral approximations of the second-order cone.
Mathematics of Operations Research, 26:193–205, 2001.

Gábor Braun, Samuel Fiorini, Sebastian Pokutta, and David Steurer.
Approximation limits of linear programs (beyond hierarchies).
Mathematics of Operations Research, 40(3):756–772, 2015.

Mehmet T. Çezik and Garud Iyengar.
Cuts for mixed 0-1 conic programming.
Mathematical Programming, 104:179–202, 2005.

Alexander Engau.
Handbook on Semidefinite, Conic and Polynomial Optimization, volume 166 of *International Series in Operations Research & Management Science*, chapter Recent Progress in Interior-Point Methods: Cutting-Plane Algorithms and Warm Starts, pages 471–498.
Springer Science+Business Media, 2012.

S. Foucart and M.-J. Lai.
Sparsest solutions of underdetermined linear systems via ℓ_q-minimization for $0 < q \leq 1$.
Applied and Computational Harmonic Analysis, 26(3):395–407, 2009.

Henrik A. Friberg.
CBLIB 2014: A benchmark library for conic mixed-integer and continuous optimization.
Mathematical Programming Computation, 8(2):191–214, 2016.

Henrik A. Friberg.
Facial reduction heuristics and the motivational example of mixed-integer conic optimization.
Technical report, Optimization-Online, 2016.

Tristan Gally and Marc E. Pfetsch.
Computing restricted isometry constants via mixed-integer semidefinite programming.
Technical report, Optimization Online, 2016.

References II

Tristan Gally, Marc E. Pfetsch, and Stefan Ulbrich.
A framework for solving mixed-integer semidefinite programs.
Optimization Methods and Software, 2017.
To Appear.

Ambros Gleixner, Leon Eifler, Tristan Gally, Gerald Gamrath, Patrick Gemander, Robert Lion Gottwald, Gregor Hendel, Christopher Hojny, Thorsten Koch, Matthias Miltenberger, Benjamin Müller, Marc E. Pfetsch, Christian Puchert, Daniel Rehfeldt, Franziska Schlösser, Felipe Serrano, Yuji Shinano, Jan Merlin Viernickel, Stefan Vigerske, Dieter Weninger, Jonas T. Witt, and Jakob Witzig.
The SCIP optimization suite 5.0.
ZIB report 17-12, 2017.

Jacek Gondzio.
Warm start of the primal-dual method applied in the cutting plane scheme.
Mathematical Programming, 83(1):125–143, 1998.

Christoph Helmberg and Franz Rendl.
Solving quadratic (0,1)-problems by semidefinite programs and cutting planes.
Mathematical Programming, 82:291–315, 1998.

IBM.
IBM ILOG CPLEX Optimization Studio.

J.E. Kelley Jr.
The cutting-plane method for solving convex programs.
Journal of the Society for Industrial and Applied Mathematics, 22:167–184, 1960.

Michal Kočvara.
Truss topology design with integer variables made easy.
Technical report, School of Mathematics, University of Birmingham, 2010.

N. Krislock, J. Malick, and F. Roupin.
Biqcrunch : a semidefinite branch-and-bound method for solving binary quadratic problems.
To appear in ACM Transactions on Mathematical Software, 2016.

M. Laurent and S. Poljak.
On a positive semidefinite relaxation of the cut polytope.
Linear Algebra and its Applications, 223/224, 1995.

J. Löfberg.
YALMIP: a toolbox for modeling and optimization in MATLAB.
In *IEEE International Symposium on Computer Aided Control Systems Design*, pages 284–289, 2004.

References III

Miles Lubin.
Mixed-integer convex optimization: outer approximation algorithms and modeling power.
PhD thesis, Massachusetts Institute of Technology, 2017.

Miles Lubin, Emre Yamangil, Russell Bent, and Juan Pablo Vielma.
Extended Formulations in Mixed-Integer Convex Programming, pages 102–113.
Springer International Publishing, Cham, 2016.

Sonja Mars.
Mixed-Integer Semidefinite Programming with an Application to Truss Topology Design.
PhD thesis, FAU Erlangen-Nürnberg, 2013.

MOSEK ApS.
The MOSEK C optimizer API manual Version 8.1 (Revision 25), 2017.

Frank Permenter, Henrik A. Friberg, and Erling D. Andersen.
Solving conic optimization problems via self-dual embedding and facial reduction: a unified approach.
Technical report, Optimization-Online, 2015.

S. Poljak and F. Rendl.
Nonpolyhedral relaxations of graph-bisection problems.
SIAM Journal on Optimization, 5(3):467–487, 1995.

I. Quesada and I.E. Grossmann.
An LP/NLP based branch and bound algorithm for convex MINLP optimization problems.
Computers & Chemical Engineering, 16(10):937–947, 1992.

Franz Rendl, Giovanni Rinaldi, and Angelika Wiegele.
Solving Max-Cut to optimality by intersecting semidefinite and polyhedral relaxations.
Mathematical Programming, 121(2):307–335, 2010.

Anders Skajaa, Erling D. Andersen, and Yinyu Ye.
Warmstarting the homogeneous and self-dual interior point method for linear and conic quadratic problems.
Mathematical Programming Computation, 5:1–25, 2013.

Makoto Yamashita, Katsuki Fujisawa, Kazuhide Nakata, Maho Nakata, Mituhiro Fukuda, Kazuhiro Kobayashi, and Kazushige Goto.
A high-performance software package for semidefinite programs: SDPA 7.
Research Report B-460, Dept. of Mathematical and Computing Science, Tokyo Institute of Technology, September 2010.