Semidefinite Programming for Power System
Stability and Optimization

MOSEK Workshop on Semidefinite Optimization in Power Flow Problems

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DTU Center for Electric Power and Energy
Outline

• Who are we? ... And what do we do?
• What is semidefinite programming (SDP) ?
• Power System Stability Assessment and SDP
• Power System Optimization and SDP
DTU Center for Electric Power and Energy

- Established 15 Aug. 2012; merger of existing units (Lyngby+Risø)
- One of the strongest university centers in Europe with ~ 100 employees

Mission: Provides cutting-edge research, education and innovation in the field of electric power and energy to meet the future needs of society regarding a reliable, cost efficient and environmentally friendly energy system

- **BSc & MSc:** Electrical Engineering, Wind Energy, Sustainable Energy
- **Direct Support from:** Energinet.dk, Siemens, DONG Energy, Danfoss

*DTU ranked world 2nd in Energy Science and Engineering*¹

¹Shanghai Ranking 2016, Global Ranking of Academic Subjects
The Energy Analytics & Markets group

One of the 5 groups of the Center for Electric Power and Energy, Department of Electrical Engineering

• Resources: (10 nationalities)
  • Faculty: 1 Prof, 2 Assist. Profs.
  • Junior: 3 post-doc fellows, 9 Ph.D. students (+2 externals), 2-3 research assistants
  • + student helpers, and Ph.D. guests from China, Brazil, US, Spain, France, Italy, Netherlands, Germany, etc.

• Projects (active in 2016):
  • EU: BestPaths
  • Danish: 5s, EcoGrid 2.0, CITIES, EnergyLab Nordhavn, EnergyBlock, CORE, MULTI-DC
  • Danish-Chinese: PROAIN

• Education: Various courses on renewables forecasting, optimization, and electricity markets

• (hopefully) recognized leading expertise in energy analytics and markets
What we really do...

Energy Analytics & Markets

- Forecasting
- Clustering & Profiling
- Big data
- Data-driven analytics

- Design
- Energy markets
- Uncertainty, variability & flexibility
- Offering strategy & trading
- Modelling & Simulation

- System models & Optimization
- Large-scale optimization
- Equilibrium models
- Stochastic optimization

- Open access datasets
- Open-source software
- Open dissemination
- Open courses
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Open access datasets
Open-source software
Open dissemination
Open courses
Research Topics (Selection)

• Optimal operation of combined heat, gas, and electricity networks
• Game theoretical approaches for electricity market participants
• Spatiotemporal forecasting for wind, solar, and energy demand
• Stochastic electricity market design and value of information
• HVDC optimization and control under uncertainty
Outline

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• What is semidefinite programming (SDP)?
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What is Semidefinite Programming? (SDP)

• SDP is the “generalized” form of an LP (linear program)

Linear Programming

\[ \min c^T \cdot x \]

subject to:

\[ a_i \cdot x = b_i, \quad i = 1, \ldots, m \]
\[ x \geq 0, \quad x \in \mathbb{R}^n \]

Semidefinite Programming

\[ \min C \bullet X := \sum_i \sum_j C_{ij} X_{ij} \]

subject to:

\[ A_i \bullet X = b_i, \quad i = 1, \ldots, m \]
\[ X \succeq 0 \]

• LP: Optimization variables in the form of a vector \( x \).

• SDP: Optim. variables in the form of a positive semidefinite matrix \( X \).

• SDP=LP: for diagonal matrices
Example: Feasible space of SDP vs LP variables

LP

\[ \begin{align*}
    x_1 & \geq 0 \\
    x_2 & \geq 0
\end{align*} \]

SDP

\[ X = \begin{bmatrix} x_2 & x_1 \\ x_1 & 1 \end{bmatrix} \succeq 0 \Rightarrow x_2 - x_1^2 \geq 0 \]
Example: Feasible space of SDP vs LP variables

LP

\[
\begin{align*}
  x_1 & \geq 0 \\
  x_2 & \geq 0
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\]

SDP

\[
X = \begin{bmatrix} x_2 & x_1 \\ x_1 & 1 \end{bmatrix} \succeq 0 \Rightarrow x_2 - x_1^2 \geq 0
\]

- In SDP we can express quadratic constraints, e.g. \( x_1^2 \) or \( x_1 x_2 \)
- optimization variables need not be strictly non-negative
- LP is a special case of SDP
SDP for Power System Stability

find a feasible $X$

subject to:

\[ A_i \bullet X \succeq 0 \]

\[ X \succeq 0 \]

SDP for Optimal Power Flow

minimize cost of electricity

\[ \min C \bullet X \]

subject to:

voltage and power flow constraints

\[ A_i \bullet X = b_i, \quad i = 1, \ldots, m \]

\[ X \succeq 0 \]
Robust Power System Stability Assessment with Extensions to Inertia and Topology Control

work with:
Thanh Long Vu, Kostya Turitsyn
MIT Mechanical Engineering
Power blackouts

Statistics:

- Frequency: \( \frac{1}{1\text{hr/year}} \)
- Economic damage: \( \approx 100\text{B\$}/\text{year} \)
- Total electric energy cost in US: \( \approx 400\text{B\$}/\text{year} \)

Challenges and opportunities:

- New algorithms for better decision-making
Power blackouts

Statistics:

- Frequency: $\approx 1\text{hr/year} \implies$ economic damage: $\approx 100B$/year
- Total electric energy cost in US: $\approx 400B$/year
Power blackouts

Statistics:

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Dynamic Security Assessment

- Security = ability to withstand disturbances

- Security Assessment:
  - Screen contingency list every 15 mins
  - Prepare contingency plans for critical scenarios.
Dynamic Security Assessment

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- Security Assessment:
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  - Prepare contingency plans for critical scenarios.

- Dynamic simulations are hard:
  - DAE system with about 10k degrees of freedom
  - Faster than real-time simulations is an open research topic

- Alternative: Energy methods = Security certificates
Security certificates

- Security region: non-convex, NP-hard characterization
Security certificates

- Security region: non-convex, NP-hard characterization
- Security certificates: tractable sufficient conditions
Security certificates

- Security region: non-convex, NP-hard characterization
- Security certificates: tractable sufficient conditions
- Strategy: certify security of most of scenarios with conservative conditions, use simulations for few really dangerous scenarios
Closest mechanical equivalent to a power system is a mass-spring system.
Energy method

- If $E(\delta_0, \dot{\delta}_0) < E_{CUEP}$, then stable
Energy method

• If $E(\delta_0, \dot{\delta}_0) < E_{CUEP}$, then stable
• Fast transient stability certificate
Energy method

- If $E(\delta_0, \dot{\delta}_0) < E_{CUEP}$, then stable
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- Computing $E_{CUEP}$ is an NP-hard problem
Energy method

- If $E(\delta_0, \dot{\delta}_0) < E_{CUEP}$, then stable
- Fast transient stability certificate
- Computing $E_{CUEP}$ is an NP-hard problem
- Certificates are generally conservative
Modeling Approach

- Non-linear swing equation

\[ m_k \ddot{\delta}_k + d_k \dot{\delta}_k + \sum_{j \in \mathcal{N}_k} a_{kj} \sin(\delta_k - \delta_j) = P_k \]  

(1)

\[ m_k \ddot{\delta}_k + d_k \dot{\delta}_k + \sum_{j \in \mathcal{N}_k} a_{kj} (\sin(\delta_{kj}) - \sin(\delta_{kj}^*)) = 0 \]  

(2)

\[ \dot{x} = Ax - BF(Cx) \]  

(3)

- Structure-preserving model: \( A \) and \( B \) do not correspond to the reduced model

- \( x = \delta_i - \delta_i^* \)

- \( A, B, C \) are independent of the operating point \( P_k \)

- \( F(Cx) \) stands for the non-linear function \( \sin(\delta_{kj}) - \sin(\delta_{kj}^*) \)
Bounding nonlinearity

\[ \left\{ k_j, \dot{\theta}_j : |k_j| < \pi/2 \right\} \]
Bounding nonlinearity

- Sector bound on nonlinearity for polytope $\mathcal{P}$: $\{\delta, \dot{\delta} : |\delta_{k,j}| < \frac{\pi}{2}\}$
Stability certificate

• If:

\[
\bar{A}^T P + P \bar{A} + \frac{(1 - g)^2}{4} C^T C + P B B^T P \leq 0
\]  

(4)

• there exists a quadratic Lyapunov function \( V = x^T P x \) that is decreasing whenever \( x(t) \in \mathcal{P} \).
Stability region

- Lyapunov function $x^T P x$ is an ellipsoid
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• Due to the sector bound on the nonlinear $\sin()$ term, stability is certified only as long as we stay within $[-\pi/2, \pi/2]$
- Lyapunov function $x^T P x$ is an ellipsoid

- Due to the sector bound on the nonlinear $\sin()$ term, stability is certified only as long as we stay within $[-\pi/2, \pi/2]$.

- Finding the $V_{min}$ within these bounds is now a convex problem!
  - We can solve (even large) convex problems fast and efficiently.
Extensions to Remedial Actions

• Can incorporate inertia and damping control by appropriately changing $A$ and $B \Rightarrow$ bound the growth of Lyapunov function

\[
\delta_0 = \delta_F(\tau_{clearing})
\]

\[
\delta_{\text{pre}}^* = \delta_{\text{post}}^* = \delta^*
\]

Fault–on trajectory

Post–fault trajectory
Extensions to Remedial Actions

• Can incorporate inertia and damping control by appropriately changing $A$ and $B \Rightarrow$ bound the growth of Lyapunov function

• Can incorporate topology control, e.g. FACTS, by appropriately changing $A$ and $B \Rightarrow$ generate a set of ellipsoids that will guarantee the convergence of $x_0$ to the post-fault equilibrium
Convex Relaxations of Chance Constrained AC Optimal Power Flow

work with:
Andreas Venzke
The Optimal Power Flow Problem (OPF)

minimize the cost of electricity generation

subject to:

demand of electric loads

maximum power of generators

maximum power capacity of transmission lines

voltage limits

• The problem is:

  • non-linear: power flow depends on the square of voltages
  • non-convex: there are more than one (local) minima
Convex vs. Non-convex Problem

Convex Problem

One global minimum

Non-convex problem

Several local minima
Several local minima: So what?

- **Electricity Markets:** Assume that the difference in the cost function of a local minimum versus a global minimum is 2%

- The total electric energy cost in the US is \( \approx 400 \) Billion\$/year

- 2% amounts to 8 billion US\$ in economic losses per year

- **Technical operation:** Convex OPF determines absolute lower or upper bound of control effort \( \rightarrow \) useful in branch-and-bound methods for mixed integer programming, e.g. unit commitment, capacitor switching

- Convex problems guarantee that we find a global minimum \( \rightarrow \) convexify the OPF problem
Convexifying the Optimal Power Flow problem (OPF)

- Convex relaxation transforms OPF to convex Semi-Definite Program (SDP)

\[ f(x) \]

Convex Relaxation

\[ x \]

Cost

Convexifying the Optimal Power Flow problem (OPF)

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\[ \tilde{f}(x) \]

Convex Relaxation

---

Convexifying the Optimal Power Flow problem (OPF)

- Convex relaxation transforms OPF to convex Semi-Definite Program (SDP)

- Under certain conditions, the obtained solution is the global optimum to the original OPF problem\(^2\)

\[ \text{Cost} \]

\[ f(x) \quad \tilde{f}(x) \]

Convex Relaxation

---

Transforming the AC-OPF to an SDP

• Power is a quadratic function of voltage, e.g.: \( P_{ij} = f(V_i^2, V_j^2, V_iV_j) \)
Transforming the AC-OPF to an SDP

- Power is a quadratic function of voltage, e.g.: $P_{ij} = f(V_i^2, V_j^2, V_i V_j)$
- Let $W = VV^T$ and express $P = f(W)$. In that case, $P$ is an affine function of $W$. 
Transforming the AC-OPF to an SDP

- Power is a quadratic function of voltage, e.g.: $P_{ij} = f(V_i^2, V_j^2, V_i V_j)$

- Let $W = V V^T$ and express $P = f(W)$. In that case, $P$ is an affine function of $W$.

- If $W \succeq 0$ and rank($W$) = 1:
  
  $W$ can be expressed as a product of vectors and we can recover the solution $V$ to our original problem

- However the rank-1 constraint is non-convex...
Applying convex relaxations with SDP

\[ f(x) \]  
\[ \tilde{f}(Y^*) \leq f(x^*) \]

\[ f(x^*) = \tilde{f}(Y^*) \]

\[ \text{rank}(Y^*) = 1 \]

EXACT: \[ W = VV^T \]

\[ \Downarrow \]

RELAX: \[ W \succeq 0 \]

\[ \text{rank}(W) = 1 \]

- For the objective functions, it holds \( \text{EXACT} \geq \text{RELAX} \)

- The RELAX problem is an SDP problem!

- If \( W^* \) happens also to be rank-1, then \( \text{EXACT} = \text{RELAX} \)!
Notes on the Convex Relaxation

- **Relaxation gap**: Difference between the solution of original non-convex, non-linear OPF and the SDP

\[ f(x) \approx \tilde{f}(x) \]

- If \( \text{rank}(W) = 1 \) or 2: solution to original OPF problem can be recovered to global optimum
- If \( \text{rank}(W) \geq 3 \): the solution \( W \) has no physical meaning (but still it is a lower bound)

Notes on the Convex Relaxation

- **Relaxation gap**: Difference between the solution of original non-convex, non-linear OPF and the SDP

- If \( \text{rank}(W) = 1 \) or \( 2 \): solution to original OPF problem can be recovered → global optimum

- If \( \text{rank}(W) \geq 3 \): the solution \( W \) has no physical meaning (but still it is a lower bound)

- Molzahn\(^3\) derives a heuristic rule: if the ratio of the 2nd to the 3rd eigenvalue of \( W \) is larger than \( 10^5 \) → we obtain rank-2.

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Introducing Uncertainty

- Increasing share of uncertain renewables
  ⇒ Include chance constraints in OPF:
  Constraints should be fulfilled for a defined probability $\epsilon$, given an underlying distribution of the uncertainty

![Probability distribution](image)
Introducing Uncertainty

- Increasing share of uncertain renewables
  ⇒ Include chance constraints in OPF:
  Constraints should be fulfilled for a defined probability \( \epsilon \), given an underlying distribution of the uncertainty

- Uncertainty in wind forecast errors

- Our Goal: Convex Chance-Constrained AC-OPF

- Pros:
  - Can consider losses and large uncertainty deviations
  - Considers reactive power → reactive power flow control
  - Convex → can find global optimum

- Cons:
  - Scalable?
Uncertainty Sets - Rectangular & Gaussian

How to model the uncertainty distribution of forecast errors $\Delta P_{W_i}$?

Rectangular uncertainty set: General non-Gaussian distributions. Upper and lower bounds are known a-priori.

Ellipsoid uncertainty set: Multivariate Gaussian distribution with known standard deviation and confidence interval.

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Uncertainty Sets - Rectangular & Gaussian

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Rectangular uncertainty set: General non-Gaussian distributions. Upper and lower bounds are known a-priori.

Ellipsoid uncertainty set: Multivariate Gaussian distribution with known standard deviation and confidence interval $\epsilon$.

- First steps taken in Vrakopoulou et al, 2013. Here we extend this work in several ways.

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Formulation for Rectangular Uncertainty Set

It suffices to enforce the chance constraints at the vertices $v$ of the uncertainty set\(^4\).

Modified IEEE 9-bus system with wind farms W1 and W2

- W1 with ± 50 MW deviation inside confidence interval
- W2 with ± 40 MW deviation inside confidence interval
- SDP-Solver: MOSEK v8
- Coded with Julia (open-source)
Simulation Results

Affine Policy for Rectangular Uncertainty Set

| Generator droops | $d_1 = [0.5 \ 0.25 \ 0.25 \ 0 \ -1 \ 0 \ 0 \ 0 \ 0]$ |
| Generator droops | $d_2 = [0.5 \ 0.25 \ 0.25 \ 0 \ 0 \ -1 \ 0 \ 0 \ 0]$ |
| Weight power loss | $\mu = 0.4 \ \frac{$}{hMW}$ |

Generator cost $= 3378.73 \ \frac{$}{h}$

Eigenvalue ratios

- $\rho(W_0) = 6.4 \times 10^6$
- $\rho^*(W_0 + \Delta P_{1}^{\max} \Delta B_1) = 2.5 \times 10^5$
- $\rho^*(W_0 + \Delta P_{2}^{\max} \Delta B_2) = 2.4 \times 10^5$
- $\rho^*(W_0 + \Delta P_{3}^{\max} \Delta B_3) = 2.7 \times 10^6$
- $\rho^*(W_0 + \Delta P_{4}^{\max} \Delta B_4) = 1.9 \times 10^6$

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<td>G1</td>
<td>1.10</td>
<td>64.70</td>
<td>8.09</td>
<td>1.07</td>
<td>60.96</td>
<td>31.00</td>
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<tr>
<td>G2</td>
<td>1.09</td>
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<td>1.39</td>
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<td>0.00</td>
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<td>317.34</td>
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<table>
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<th># Branch</th>
<th>from</th>
<th>to</th>
<th>$P_{lm}$ [MW]</th>
<th>$P_{lm}^*$ [MW]</th>
<th>$Q_{lm}$ [Mvar]</th>
<th>$Q_{lm}^*$ [Mvar]</th>
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<tr>
<td>3</td>
<td>5</td>
<td>6</td>
<td>42.87</td>
<td>67.50</td>
<td>-24.07</td>
<td>-35.04</td>
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Maximum voltage [p.u.] $= V_{\text{max}} = 1.100$, $(V_{\text{max}}^*) = 1.100$

- we satisfy the conditions to obtain the global optimum
- all constraints are satisfied
- we find the true global minimum
Ongoing Work

- Convex formulation for chance-constrained AC-OPF
- Investigating the conditions to obtain zero relaxation gap
- Investigating how to achieve scalability
- Extending this formulation to combined AC and HVDC grids

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Conclusions

• “Semidefinite programming is the most exciting development in mathematical programming in the 1990’s”\textsuperscript{6}

• Power interruptions are extremely costly; secure operation is challenging
  • SDP-based methods can extract less conservative stability certificates
• Large systems have high costs \Rightarrow cannot afford to find a suboptimal local minimum
  • SDP-based optimization allows to recover the global optimum
  • We introduced convex relaxations for a chance-constrained AC-OPF

• Challenges: Numerics & scalability

Thank you!

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References: