Numerical Aspects of Semidefinite Relaxations of Optimal Power Flow Problems

MOSEK workshop on semidefinite optimization in power flow problems

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Electric power systems

- generators (supply)
- transmission lines
- transformers
- consumers (demand)
- market (price mechanism)
- stability (physics)
- laws & regulations

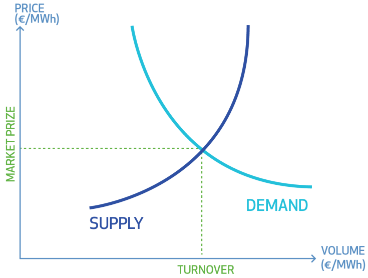
Power flow optimization

Multifaceted problem

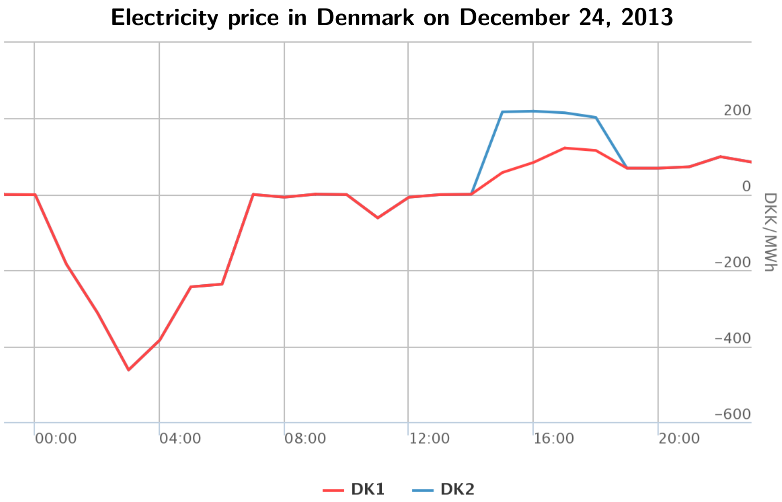
- economically: market equilibrium (nonlinear pricing)
- electrically: physics (stability and power balance)
- computationally: large-scale, nonconvex, binary variables
- robustness: withstand loss of generating unit(s) and/or transmission line(s)
- planning purposes: investment in new equipment, maintenance, etc.
- forecasts – supply, demand, weather, commodity prices

The electricity market

- Deregulation in the early 1990s
- Nord Pool Spot
    - Scandinavia, Germany, Baltics, UK
- Day-ahead market (Elspot)
    - hour by hour bids for next day
    - unit commitment problem
- Intraday market (Elbas)
    - trading until one hour before delivery
    - helps secure balance
    - increasingly important: wind, solar, ...
    - OPF problem “solved” every 5 min.



Negative prices



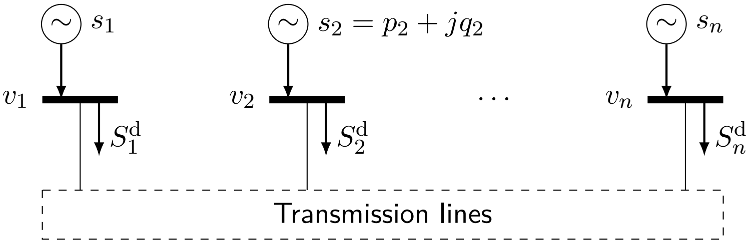
The War of Currents (1880s)

Edison vs. Westinghouse — DC vs. AC distribution network

AC systems

- complex power $=$ real/active power $+\ j\cdot$reactive power
- apparent power $=$ magnitude of complex power
- impedance $=$ resistance $+\ j\cdot$reactance
- admittance $=$ (impedance)$^{-1}$ $=$ conductance $+\ j\cdot$susceptance

Power network model

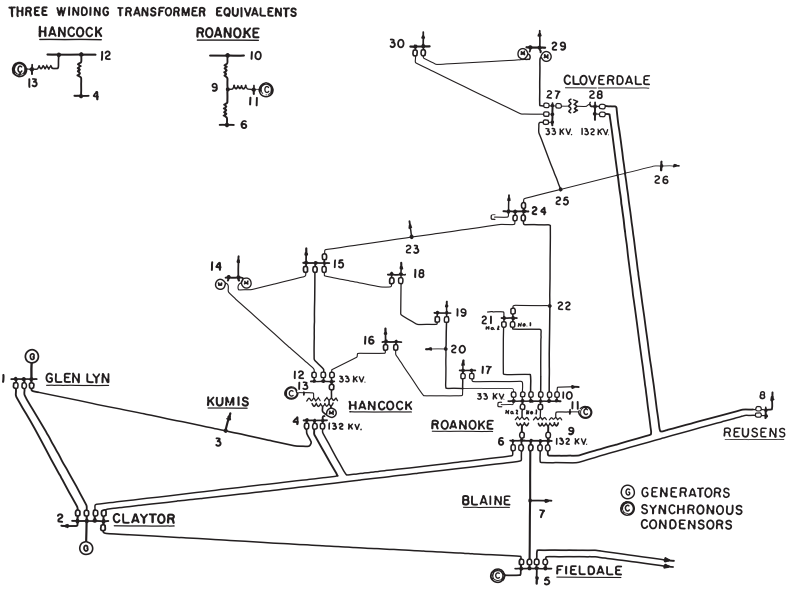


Kirchhoff's current law

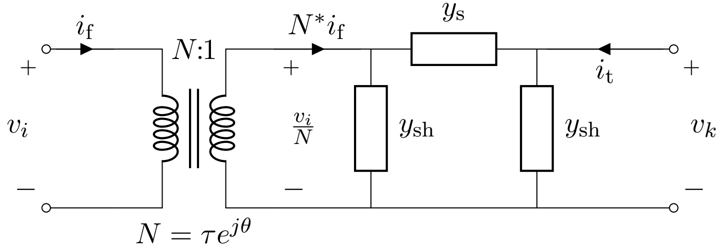
$$i_k = \bar{y}_k v_k + \sum_{i \in \mathrm{adj}(k)} y_{ki}(v_k - v_i)$$

$$i = Yv$$

Example of power network: IEEE30



Transmission line model



$$\begin{bmatrix} i_{\mathrm{f}} \\ i_{\mathrm{t}} \end{bmatrix} = \begin{bmatrix} (y_{\mathrm{s}} + y_{\mathrm{sh}}) \frac{1}{NN^*} & -y_{\mathrm{s}} \frac{1}{N^*} \\ -y_{\mathrm{s}} \frac{1}{N} & y_{\mathrm{s}} + y_{\mathrm{sh}} \end{bmatrix} \begin{bmatrix} v_i \\ v_k \end{bmatrix}$$

Complex power injections can be expressed as

$$S_{i,k} = v_i i_{\mathrm{f}}^* = v^H \widetilde{Y}_{\mathrm{f}} v, \quad S_{k,i} = v_k i_{\mathrm{t}}^* = v^H \widetilde{Y}_{\mathrm{t}} v$$

Classical optimal power flow problem (I)

$$
\begin{array}{ll}
\text{minimize} & \sum_{i\in\mathcal{G}} a_i(p_i)^2 + b_i p_i + c_i \\
\text{subject to} & v_k^{\min} \le |v_k| \le v_k^{\max},\ \ k = 1,\dots,n \\
& i_k^* v_k = \sum_{i\in\mathcal{G}_k} s_i - S_k^{\mathrm{d}},\ \ k = 1,\dots,n \\
& S_i^{\min} \preceq s_i \preceq S_i^{\max},\ \ i \in \mathcal{G} \\
& |S_{k,l}| \le S_{k,l}^{\max}, |S_{l,k}| \le S_{l,k}^{\max},\ \ (k,l) \in \mathcal{L}
\end{array}
$$

- studied in the 1960s by Carpentier (*economic dispatch problem*) (Carpentier 1962)
- variables are voltages $v \in \mathbb{C}^n$ and generation power $s_i = p_i + jq_i$, $i \in \mathcal{G}$
- eliminate currents: $i_k = e_k^T Y v$
- objective: minimize production cost, loss, ...

Classical optimal power flow problem (II)

Non-convex quadratic problem

$$
\begin{array}{ll}
\text{minimize} & \sum_{i\in\mathcal{G}} a_i(p_i)^2 + b_i p_i + c_i \\
\text{subject to} & (v_k^{\min})^2 \le v^H(e_k e_k^T)v \le (v_k^{\max})^2,\ k=1,\dots,n \\
& v^H Y_k v = \sum_{i\in\mathcal{G}_k} s_i - S_k^{\mathrm{d}},\ k=1,\dots,n \\
& S_i^{\min} \preceq s_i \preceq S_i^{\max},\ i\in\mathcal{G} \\
& |v^H T_{k,l} v| \le S_{k,l}^{\max}, |v^H T_{l,k} v| \le S_{l,k}^{\max},\ (k,l)\in\mathcal{L}
\end{array}
$$

- nonlinear optimization: SQP, Lagrangian relaxation, and interior-point methods
- derivative-free methods: particle swarm optimization, genetic algorithms, and evolutionary programming
- dual variables have valuable economic meanings in electricity markets
- semidefinite relaxation (Bai et al. 2008; Lavaei et al. 2012)

Reformulation

- epigraph formulation $a_i(p_i)^2 + b_i p_i \leq t_i$ equilvalent to SOC constraint

$$\begin{bmatrix} 1/2 + t_i - b_i p_i \\ 1/2 - t_i + b_i p_i \\ \sqrt{2a_i} p_i \end{bmatrix} \in \mathcal{Q}_3$$

- nonnegative voltage magnitude slack variables

$$(v_k^{\min})^2 + \nu_k^l = |v_k|^2 \qquad |v_k|^2 + \nu_k^u = (v_k^{\max})^2$$

- nonnegative power slack variables

$$S_i^{\min} + s_i^l = s_i \qquad s_i + s_i^u = S_i^{\max}$$

- flow constraint $|v^H T_{k,l} v| \leq S_{k,l}^{\max}$ equivalent to SOC constraint

Semidefinite relaxation of QCQP

Non-convex quadratically constrained quadratic program

$$\begin{array}{ll} \text{minimize} & x^T A_0 x + 2b_0^T x + c_0 \\ \text{subject to} & x^T A_i x + 2b_i^T x + c_i \leq 0, \quad i = 1, \ldots, m \end{array}$$

variable $x \in \mathbb{R}^n$

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(A_0 X) + 2b_0^T x + c_0 \\ \text{subject to} & \mathbf{tr}(A_i X) + 2b_i^T x + c_i \leq 0, \quad i = 1, \ldots, m \\ & X = xx^T \end{array}$$

Semidefinite relaxation replace $X = xx^T$ by

$$X \succeq xx^T \;\Leftrightarrow\; \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0$$

(Lovász 1979; Shor 1987; Goemans et al. 1995)

Semidefinite optimization

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(CX) \\ \text{subject to} & \mathbf{tr}(A_iX) = b_i,\ i = 1, \ldots, m \\ & X \succeq 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & \sum_{i=1}^m A_i y_i + S = C \\ & S \succeq 0 \end{array}$$

Exploiting sparsity

- form Schur complement equations efficiently: $H_{ij} = \mathbf{tr}(A_i W_1 A_j W_2)$
  (Fujisawa et al. 1997)
- matrix completion
  (Fukuda et al. 2001; Nakata et al. 2003; Burer 2003; Andersen et al. 2010)
- chordal conversion
  (Fukuda et al. 2001; Nakata et al. 2003; Kim et al. 2010)

Minimum semidefinite rank

optimality condition $XS = 0$ implies that

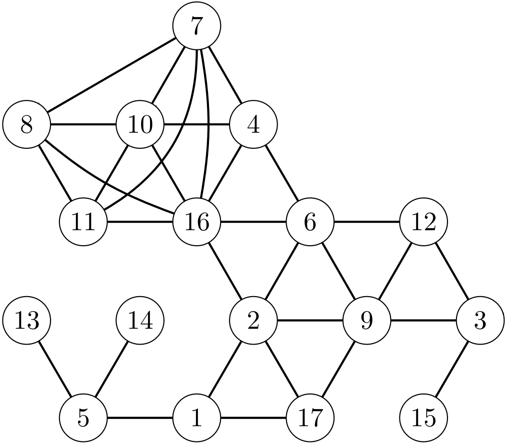
$$\mathbf{rank}(X) + \mathbf{rank}(S) \leq n$$

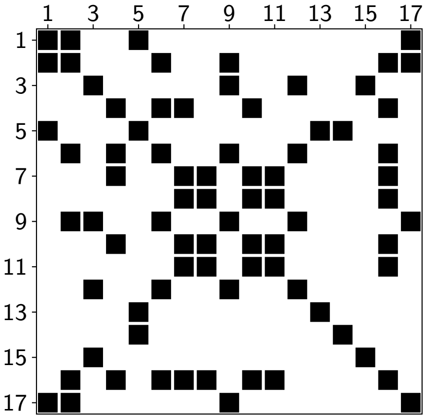
minimum semidefinite rank of simple connected graph G

$$\mathrm{msr}(G) = \min\{\mathbf{rank}(S) \,|\, S \in \mathbb{S}^n_{E,+},\ S_{ij} \neq 0\ \forall (i,j) \in E\}$$

- G is a tree: $\mathrm{msr}(G) = n - 1$
- induced subgraph $G(W)$: $\mathrm{msr}(G(W)) \leq \mathrm{msr}(G)$
- lower-bound based on tree-size of G: $\mathrm{ts}(G) - 1 \leq \mathrm{msr}(G)$
- $\mathrm{msr}(G) \leq \mathrm{cc}(G)$ (equality if G is chordal)

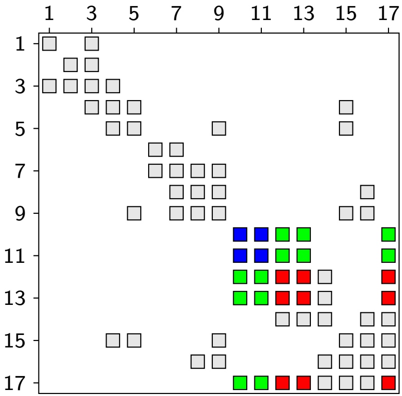
Chordal graphs and matrices

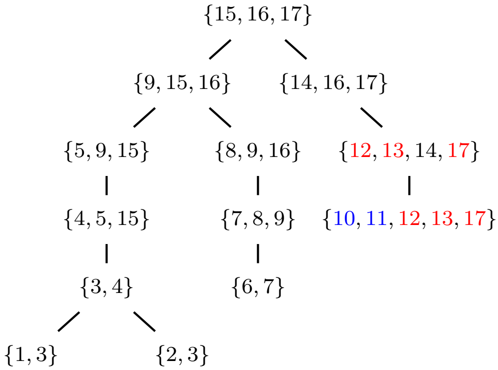




a graph is chordal if all cycles of length ≥ 4 have a chord

Cliques and clique tree





running intersection property (induced subtree property)

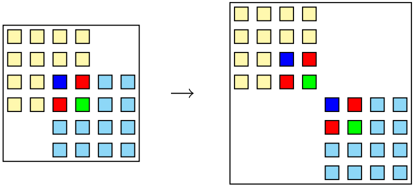
$$W \cap W' \neq \emptyset \quad \Leftrightarrow \quad W \cap W' \subseteq \text{ cliques on path between } W \text{ and } W'$$

Cone of positive semidefinite completable matrices

Theorem (Grone et al. 1984)
$\mathbb{S}^n_{E,\mathrm{c}} \equiv$ set of partial nonnegative matrices in $\mathbb{S}^n_E$ iff E is chordal

conversion to block-diagonal structure (Fukuda et al. 2001)

$$X \in \mathbb{S}^n_{E,\mathrm{c}} \quad \Leftrightarrow \quad \begin{cases} \hat{X} = \mathbf{diag}(X_1, \ldots, X_l) \succeq 0 \\ \mathbf{tr}(B_i \hat{X}) = 0, \; i = 1, \ldots, M \end{cases}$$



equality constraints: $M = \sum_i k_i(k_i+1)/2, \quad k_i = |W_i \cap W_{\mathrm{par}(i)}|$

Second-order cone relaxation

2×2 matrix for each transmission line $(i, j) \in E$

$$X^{(i,j)} = \begin{bmatrix} X_{11}^{(i,j)} & X_{12}^{(i,j)} \\ X_{21}^{(i,j)} & X_{22}^{(i,j)} \end{bmatrix} = \begin{bmatrix} x_i \\ x_j \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix}^H$$

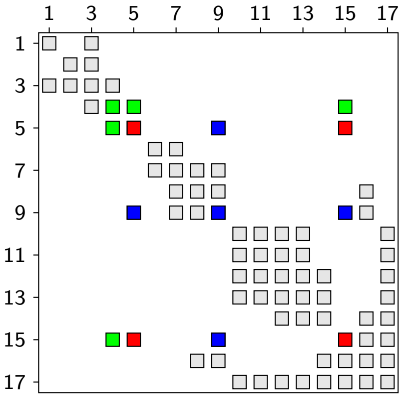
express $X^{(i,j)} \succeq 0$ as rotated second-order cone constraint

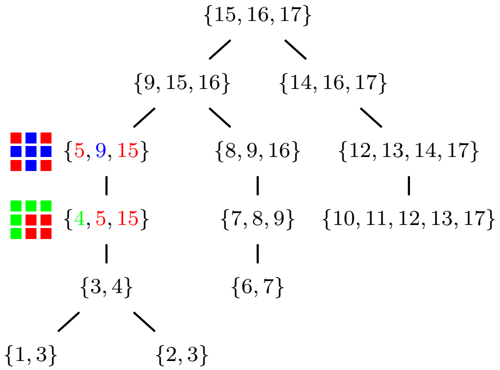
$$X_{11}^{(i,j)} X_{22}^{(i,j)} \geq |X_{21}^{(i,j)}|^2$$

- generally weaker than the semidefinite relaxation
- equivalent to the semidefinite relaxation when the network is a tree

Clique amalgamation

- merge cliques to reduce number of coupling constraints
- fewer but larger cliques





Clique amalgamation

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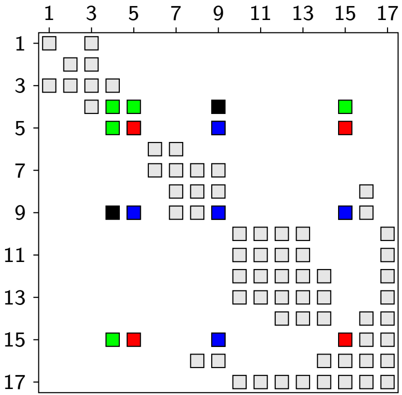

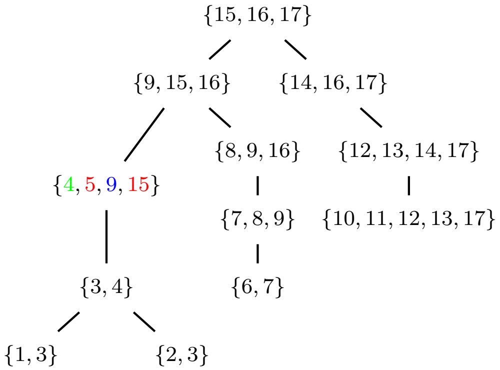

Conversion-based relaxations

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(CX) \\ \text{subject to} & \mathbf{tr}(A_i X) = b_i, \quad i = 1, \ldots, m \\ & X \succeq 0 \end{array}$$

chordal embedding with l cliques

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(\hat{C}\hat{X}) \\ \text{subject to} & \mathbf{tr}(\hat{A}_i \hat{X}) = b_i, \quad i = 1, \ldots, m \\ & \mathbf{tr}(B_i \hat{X}) = 0, \quad i \in \mathcal{I} \\ & \hat{X} = \mathbf{diag}(X_1, ..., X_l) \succeq 0 \end{array}$$

Numerical experiments

Test cases

- 38 test cases from Matpower 6.0 (with $r_{\min} = 10^{-5}$) (Zimmerman et al. 2011)
- 20 (out of 38) test cases have 1000+ busses

SDPs

- Complex-valued SDPs "assembled" explicitly using Python/CVXOPT (Andersen et al. 2016)
- SDP conversion and complex-to-real via Python/Chompack (Andersen and Vandenberghe 2016)
- 38×3 (conversions) $\times\ 2$ (unscaled, scaled) $= 228$ SDPs

Solvers

- MOSEK 8: `cvxopt.msk.conelp`, default parameters
- SeDuMI 1.3: default param., except $\epsilon = 10^{-7}$ (10^{-8}) and max 300 iter. (150)

Small test cases (up to 300 busses)

Case	m	n	lin	soc	psd	smax	smed
case5	142	42	30	4	1	10	10
case6ww	246	105	27	25	1	12	12
case9	288	120	33	21	7	6	6
case9Q	288	120	33	21	7	6	6
case9target	288	120	33	21	7	6	6
case14	480	122	53	5	12	6	6
case24_ieee_rts	1398	553	200	98	20	10	6
case30pwl	1262	471	84	82	26	8	6
case30Q	1286	489	90	88	26	8	6
case_ieee30	1040	243	90	6	26	8	6
case30	1286	489	90	88	26	8	6
case39	1578	595	128	102	34	8	6
case57	2766	581	149	7	52	12	6
case89pegase	8280	2027	226	154	77	24	6
case118	5580	1320	506	54	108	10	6
case145	12014	2707	540	166	111	22	8
case_illinois200	10590	3486	583	521	189	18	6
case300	13724	3070	945	69	278	14	6

Small test cases (up to 300 busses)

Case	× m		× n		× psd		× smax		× smed	
	8	16	8	16	8	16	8	16	8	16
case5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
case6ww	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
case9	1.46	1.46	0.88	0.88	0.14	0.14	3.00	3.00	3.00	3.00
case9Q	1.46	1.46	0.88	0.88	0.14	0.14	3.00	3.00	3.00	3.00
case9target	1.46	1.46	0.88	0.88	0.14	0.14	3.00	3.00	3.00	3.00
case14	1.36	1.77	0.70	0.66	0.17	0.08	3.67	4.67	2.67	4.67
case24_ieee_rts	1.41	1.72	0.84	0.84	0.10	0.10	3.00	4.20	4.50	4.50
case30	1.45	2.92	0.86	0.82	0.23	0.08	3.25	7.25	1.50	5.33
case30Q	1.45	2.92	0.86	0.82	0.23	0.08	3.25	7.25	1.50	5.33
case30pwl	1.46	2.96	0.86	0.81	0.23	0.08	3.25	7.25	1.50	5.33
case_ieee30	1.56	3.37	0.72	0.63	0.23	0.08	3.25	7.25	1.50	5.33
case39	1.84	2.84	0.85	0.83	0.12	0.06	4.00	7.25	4.33	7.00
case57	1.69	2.10	0.55	0.51	0.10	0.06	3.17	3.83	5.00	7.00
case89pegase	1.19	1.57	0.65	0.56	0.18	0.06	1.67	2.42	4.50	7.67
case118	1.57	1.95	0.64	0.62	0.11	0.08	4.00	4.60	4.00	6.33
case145	1.11	1.40	0.58	0.49	0.18	0.10	1.91	2.91	3.00	5.25
case_illinois200	1.66	2.44	0.78	0.75	0.11	0.06	2.22	4.00	4.67	7.67
case300	1.52	2.21	0.62	0.57	0.17	0.07	2.43	3.57	3.50	6.83

Large test cases (1000+ busses)

Complex-valued SDR $\rightarrow$ conversion $\rightarrow$ complex-to-real

Case	m	n	lin	soc	psd	smax	smed
case1354pegase	67092	21402	3748	2864	1287	26	6
case1888rte	85306	28710	4918	4152	1816	26	4
case1951rte	87392	29362	5346	4198	1879	28	4
case2383wp	160948	47178	5812	5792	2312	50	6
case2736sp	185294	53774	6080	6538	2652	50	6
case2737sop	184042	53472	5916	6538	2653	48	6
case2746wp	185730	53937	6412	6558	2659	48	6
case2746wop	191810	55100	6312	6614	2653	52	6
case2848rte	129746	39163	7612	4462	2739	36	4
case2868rte	130986	39750	7852	4562	2763	34	4
case2869pegase	172584	49112	7778	5486	2700	30	6
case3012wp	207610	59855	7310	7132	2916	56	6
case3120sp	216574	62318	7168	7362	3029	54	6
case3375wp	231592	64746	8404	7132	3248	60	6
case6468rte	342778	82986	14420	4626	6153	60	6
case6470rte	347534	88062	15510	6220	6149	60	6
case6495rte	345518	87623	15532	6218	6171	62	6
case6515rte	346308	87925	15586	6262	6193	62	6
case9241pegase	693752	170275	24262	12590	8577	70	6
case13659pegase	741106	158960	43686	0	12997	70	6

Large test cases (1000+ busses)

Complex-valued SDR → conversion with clique merging → complex-to-real

Case	× m		× n		× psd		× smax		× smed	
	8	16	8	16	8	16	8	16	8	16
case1354pegase	1.72	2.62	0.81	0.77	0.16	0.08	1.77	3.00	3.67	7.17
case1888rte	1.76	2.64	0.85	0.80	0.18	0.09	1.77	2.69	4.00	9.50
case1951rte	1.81	2.72	0.84	0.80	0.18	0.08	1.57	2.64	4.00	10.00
case2383wp	1.53	2.21	0.76	0.71	0.12	0.06	1.24	1.68	4.33	7.67
case2736sp	1.50	2.14	0.76	0.71	0.13	0.07	1.20	1.88	4.33	7.67
case2737sop	1.50	2.13	0.75	0.71	0.13	0.07	1.21	1.79	4.33	7.67
case2746wp	1.50	2.12	0.76	0.71	0.12	0.07	1.29	1.96	4.33	7.50
case2746wop	1.49	2.08	0.75	0.70	0.13	0.07	1.15	1.69	4.33	7.67
case2848rte	1.82	2.78	0.82	0.77	0.17	0.08	1.56	2.33	5.50	10.50
case2868rte	1.79	2.75	0.81	0.77	0.17	0.08	1.53	2.29	5.00	10.00
case2869pegase	1.53	2.24	0.75	0.69	0.15	0.07	1.60	2.60	4.00	7.33
case3012wp	1.52	2.15	0.77	0.71	0.13	0.07	1.14	1.50	4.33	7.33
case3120sp	1.51	2.12	0.77	0.71	0.13	0.07	1.19	1.70	4.00	7.33
case3375wp	1.53	2.18	0.77	0.72	0.13	0.07	1.10	1.37	4.33	7.67
case6468rte	1.66	2.52	0.74	0.68	0.16	0.08	1.10	1.47	3.67	7.00
case6470rte	1.66	2.48	0.76	0.70	0.16	0.08	1.10	1.67	3.67	7.00
case6495rte	1.67	2.52	0.76	0.70	0.16	0.08	1.10	1.61	3.67	7.00
case6515rte	1.67	2.51	0.76	0.70	0.16	0.08	1.10	1.61	3.67	7.00
case9241pegase	1.38	1.94	0.68	0.60	0.14	0.07	1.03	1.49	4.33	7.67
case13659pegase	1.62	2.42	0.68	0.60	0.18	0.09	1.11	1.57	3.33	6.67

Solver status

All test cases

Status	MOSEK	SeDuMi
Optimal	214	76
Near optimal	2	13
Prim. infeas. cer.	5	
Near prim. infeas. cer.	7	
Max iterations		18
Unknown		121

Matpower 6.0 “solves” 28 out of 38 test cases

Solver status

Large test cases (1000+ busses)

Status	MOSEK	SeDuMi
Optimal	119	
Near optimal	1	6
Max iterations		18
Unknown		96

Matpower 6.0 “solves” 11 out of 20 test cases

MOSEK: CPU time (seconds)

Case	No merging		Merging (8)		Merging (16)	
	unscaled	scaled	unscaled	scaled	unscaled	scaled
case1354pegase	**16.1**	-1.4	-1.9	-2.8	0.8	1.1
case1888rte	**25.6**	-0.4	-1.9	-2.4	2.7	4.8
case1951rte	**29.7**	-1.1	-2.0	-2.3	5.5	4.7
case2383wp	**100.1**	-0.7	-12.4	-18.1	-0.3	2.0
case2736sp	**113.2**	-4.8	-9.8	-16.4	10.6	-3.4
case2737sop	**107.4**	-7.1	-12.7	-20.9	4.8	-5.2
case2746wp	**113.4**	-6.1	-13.4	-23.4	3.9	-2.0
case2746wop	**122.9**	-11.7	-16.7	-28.0	-2.2	-10.2
case2848rte	**42.5**	-0.5	-2.6	-3.3	5.5	8.4
case2868rte	**45.7**	-1.7	-3.5	-5.5	6.1	4.2
case2869pegase	**50.3**	-2.2	-7.4	-10.0	-0.8	-0.8
case3012wp	**152.3**	2.8	-18.3	-21.7	0.0	-0.3
case3120sp	**173.0**	1.2	-19.3	-22.2	7.4	9.2
case3375wp	**139.3**	9.9	-13.7	-10.5	-0.3	7.4
case6468rte	**159.6**	-10.6	-22.2	-30.5	1.2	1.9
case6470rte	**187.7**	-2.0	-33.0	-34.9	-9.1	-6.6
case6495rte	**193.4**	4.6	-27.1	-25.9	-3.0	-5.1
case6515rte	**193.2**	0.6	-24.9	-38.1	-0.5	-7.9
case9241pegase	**427.1**	-7.3	-102.3	-109.1	-74.5	-90.2
case13659pegase	**247.3**	-38.3	-84.6	-100.1	-71.7	-75.0

MOSEK: minimum eigenvalue ratio

Case	No merging		Merging (8)		Merging (16)	
	MOSEK	SeDuMi	MOSEK	SeDuMi	MOSEK	SeDuMi
case5	1.5e+02	1.5e+02	1.5e+02	1.5e+02	1.5e+02	1.5e+02
case6ww	3.1e+09	7.3e+09	3.1e+09	7.3e+09	3.1e+09	7.3e+09
case9	1.1e+07	4.8e+07	1.0e+07	2.0e+07	1.0e+07	2.0e+07
case9Q	1.1e+07	4.8e+07	1.0e+07	2.0e+07	1.0e+07	2.0e+07
case9target	5.1e+01		6.1e+01		6.1e+01	
case14	3.2e+06	1.3e+07	3.3e+06	1.8e+07	3.1e+06	2.5e+07
case24_ieee_rts	4.8e+06	6.2e+08	8.4e+07	4.8e+08	2.9e+07	2.5e+09
case30	1.8e+05	2.1e+02	1.2e+06	3.1e+02	2.8e+06	6.8e+02
case30Q	1.8e+05	2.1e+02	1.2e+06	3.1e+02	2.8e+06	6.8e+02
case30pwl	1.0e+02	6.4e+01	1.4e+02	1.3e+02	1.9e+02	1.5e+02
case_ieee30	5.9e+06	2.6e+07	1.2e+06	1.7e+07	3.9e+06	5.9e+07
case39	4.0e+02	4.0e+02	1.4e+03	1.4e+03	1.4e+03	1.4e+03
case57	1.7e+06	4.2e+06	4.6e+05	1.5e+07	3.2e+06	1.8e+08
case89pegase	5.8e+05	7.1e+06	1.3e+06	9.0e+06	3.6e+06	3.5e+04
case118	4.2e+02	4.2e+02	4.6e+02	4.6e+02	6.3e+02	6.3e+02
case145	2.8e+00		5.6e+00		1.0e+01	
case_illinois200	1.8e+06	1.2e+00	8.1e+06	2.6e+00	2.9e+06	6.4e+01
case300	9.6e+01	9.6e+01	2.7e+02	2.7e+02	4.8e+02	4.8e+02

MOSEK: minimum eigenvalue ratio

Case	No merging		Merging (8)		Merging (16)	
	unscaled	scaled	unscaled	scaled	unscaled	scaled
case1354pegase	**134.0**	1.0	2.9	2.9	3.8	3.8
case1888rte	**14.7**	1.1	3.3	3.7	6.1	9.1
case1951rte	**24.7**	1.0	2.2	2.6	7.3	8.3
case2383wp	**318.9**	1.0	1.8	1.7	1.7	1.6
case2736sp	**40197.7**	1.4	3.7	7.6	2.9	10.2
case2737sop	**19224.5**	1.0	2.1	2.1	3.6	3.8
case2746wp	**31822.0**	1.2	2.8	1.8	3.2	3.4
case2746wop	**15556.7**	0.9	5.3	2.6	2.5	2.4
case2848rte	**18.9**	0.9	2.3	2.6	3.1	4.3
case2868rte	**29.3**	1.0	1.6	2.2	3.4	3.0
case2869pegase	**155.3**	1.0	2.4	2.4	3.2	3.2
case3012wp	**17.7**	1.0	6.4	6.4	10.9	10.8
case3120sp	**66.2**	1.0	2.3	2.3	2.3	2.4
case3375wp	**17.1**	1.0	6.5	6.5	11.2	11.2
case6468rte	**13.2**	1.0	1.3	1.3	1.3	1.4
case6470rte	**22.5**	1.0	2.4	3.0	4.7	6.0
case6495rte	**32.4**	1.0	0.9	1.0	1.5	1.5
case6515rte	**36.2**	1.0	1.2	1.2	1.4	1.4
case9241pegase	**33.8**	1.0	2.3	2.3	3.1	3.1
case13659pegase	**27.4**	1.0	1.0	1.0	1.0	1.0

MOSEK: number of eigenvalue ratios $< 10^5$

Case	No merging		Merging (8)		Merging (16)	
	unscaled	scaled	unscaled	scaled	unscaled	scaled
case5	1	1	1	1	1	1
case6ww	0	0	0	0	0	0
case9	0	0	0	0	0	0
case9Q	0	0	0	0	0	0
case9target	7	7	1	1	1	1
case14	0	0	0	0	0	0
case24_ieee_rts	0	0	0	0	0	0
case30	0	0	0	0	0	0
case30Q	0	0	0	0	0	0
case30pwl	26	26	6	6	2	2
case_ieee30	0	0	0	0	0	0
case39	1	1	1	1	1	1
case57	0	0	0	0	0	0
case89pegase	0	0	0	0	0	0
case118	23	23	7	7	6	6
case145	111	109	20	20	11	11
case_illinois200	0	0	0	0	0	0
case300	3	4	4	2	2	2

MOSEK: number of eigenvalue ratios $< 10^5$

Case	No merging		Merging (8)		Merging (16)	
	unscaled	scaled	unscaled	scaled	unscaled	scaled
case1354pegase	6	10	3	4	2	2
case1888rte	426	297	109	84	76	58
case1951rte	470	349	128	104	68	44
case2383wp	973	970	116	158	82	69
case2736sp	4	21	0	0	0	0
case2737sop	22	19	1	1	1	1
case2746wp	81	67	1	5	0	0
case2746wop	9	38	6	1	2	2
case2848rte	277	376	168	93	100	73
case2868rte	364	370	135	95	77	78
case2869pegase	70	48	15	15	11	11
case3012wp	377	563	116	115	52	42
case3120sp	19	652	75	17	28	36
case3375wp	1359	495	186	192	90	42
case6468rte	690	577	190	183	129	98
case6470rte	720	470	257	153	153	95
case6495rte	911	909	319	267	174	167
case6515rte	909	909	330	291	197	192
case9241pegase	509	657	146	167	90	94
case13659pegase	1085	1084	288	288	164	155

MOSEK: percentage of eigenvalue ratios $< 10^5$

Case	No merging		Merging (8)		Merging (16)	
	unscaled	scaled	unscaled	scaled	unscaled	scaled
case5	100.0	100.0	100.0	100.0	100.0	100.0
case6ww	0.0	0.0	0.0	0.0	0.0	0.0
case9target	100.0	100.0	100.0	100.0	100.0	100.0
case9Q	0.0	0.0	0.0	0.0	0.0	0.0
case9	0.0	0.0	0.0	0.0	0.0	0.0
case14	0.0	0.0	0.0	0.0	0.0	0.0
case24_ieee_rts	0.0	0.0	0.0	0.0	0.0	0.0
case30pwl	100.0	100.0	100.0	100.0	100.0	100.0
case30Q	0.0	0.0	0.0	0.0	0.0	0.0
case_ieee30	0.0	0.0	0.0	0.0	0.0	0.0
case30	0.0	0.0	0.0	0.0	0.0	0.0
case39	2.9	2.9	25.0	25.0	50.0	50.0
case57	0.0	0.0	0.0	0.0	0.0	0.0
case89pegase	0.0	0.0	0.0	0.0	0.0	0.0
case118	21.3	21.3	58.3	58.3	66.7	66.7
case145	100.0	98.2	100.0	100.0	100.0	100.0
case_illinois200	0.0	0.0	0.0	0.0	0.0	0.0
case300	1.1	1.4	8.7	4.3	10.0	10.0

MOSEK: percentage of eigenvalue ratios $< 10^5$

Case	No merging		Merging (8)		Merging (16)	
	unscaled	scaled	unscaled	scaled	unscaled	scaled
case1354pegase	0.5	0.8	1.5	2.0	2.0	2.0
case1888rte	23.5	16.4	33.4	25.8	47.8	36.5
case1951rte	25.0	18.6	38.9	31.6	43.6	28.2
case2383wp	42.1	42.0	40.8	55.6	56.6	47.6
case2736sp	0.2	0.8	0.0	0.0	0.0	0.0
case2737sop	0.8	0.7	0.3	0.3	0.6	0.6
case2746wp	3.0	2.5	0.3	1.5	0.0	0.0
case2746wop	0.3	1.4	1.8	0.3	1.1	1.1
case2848rte	10.1	13.7	36.4	20.1	46.9	34.3
case2868rte	13.2	13.4	28.7	20.2	33.3	33.8
case2869pegase	2.6	1.8	3.8	3.8	6.0	6.0
case3012wp	12.9	19.3	31.0	30.7	26.0	21.0
case3120sp	0.6	21.5	19.1	4.3	13.5	17.3
case3375wp	41.8	15.2	43.8	45.2	42.3	19.7
case6468rte	11.2	9.4	19.3	18.6	26.2	19.9
case6470rte	11.7	7.6	26.0	15.5	30.9	19.2
case6495rte	14.8	14.7	32.0	26.8	35.9	34.4
case6515rte	14.7	14.7	33.1	29.2	40.0	38.9
case9241pegase	5.9	7.7	12.3	14.1	15.8	16.5
case13659pegase	8.3	8.3	12.5	12.5	14.2	13.5

Objective value, lower bound ($/hour)

Case	No merging		Merging (8)		Merging (16)	
	MOSEK	SeDuMi	MOSEK	SeDuMi	MOSEK	SeDuMi
case5	16635.77	16635.78	16635.77	16635.78	16635.77	16635.78
case6ww	3143.97	3143.97	3143.97	3143.97	3143.97	3143.97
case9	5296.76	5296.76	5296.76	5296.76	5296.76	5296.76
case9Q	5296.76	5296.76	5296.76	5296.76	5296.76	5296.76
case9target	–	100.00	–	100.00	–	100.00
case14	8081.54	8081.54	8081.54	8081.54	8081.53	8081.54
case24_ieee_rts	63352.05	63352.20	63352.18	63352.20	63352.11	63352.20
case30	576.89	x	576.89	x	576.89	x
case30Q	576.89	x	576.89	x	576.89	x
case30pwl	0.00	x	0.00	x	0.00	x
case_ieee30	8906.16	8906.16	8906.15	8906.16	8906.14	8906.16
case39	41865.06	41865.06	41865.06	41865.06	41865.06	41865.06
case57	41737.83	41737.83	41737.78	41737.83	41737.82	41737.83
case89pegase	5819.85	5819.86	5819.83	5819.85	5819.84	x
case118	129660.16	129659.89	129660.14	129660.21	129660.07	129660.20
case145	–	100.00	–	100.00	–	100.00
case_illinois200	36748.35	x	36748.36	x	36748.24	x
case300	719743.64	719743.56	719743.17	719743.72	719743.67	719743.72

Objective value, lower bound ($/hour) [MOSEK]

Case	No merging		Merging (8)		Merging (16)	
	unscaled	scaled	unscaled	scaled	unscaled	scaled
case1354pegase	**74061.23**	-1.43	-0.69	-1.78	-0.29	-0.41
case1888rte	**59603.22**	1.72	0.64	3.42	0.75	3.44
case1951rte	**81735.44**	0.83	-1.96	0.86	-0.29	2.23
case2383wp	**1858028.93**	662.25	1946.63	1503.31	3652.51	1514.65
case2736sp	**1307508.64**	75.47	49.78	226.63	-37.10	233.27
case2737sop	**777491.09**	13.92	-11.37	-35.58	-23.99	-53.44
case2746wp	**1630904.54**	186.85	402.92	206.71	351.12	400.69
case2746wop	**1208126.48**	-35.46	82.43	7.28	-193.45	-183.42
case2848rte	**53008.28**	1.84	-2.94	1.68	-2.98	1.73
case2868rte	**79796.80**	-0.06	-1.77	0.69	0.64	0.38
case2869pegase	**133990.01**	-1.83	-0.27	-3.18	-0.32	-3.09
case3012wp	**2587536.11**	158.16	-1242.14	-1199.13	-675.73	-1054.38
case3120sp	**2140694.65**	-267.28	1149.63	720.70	444.72	80.32
case3375wp	**7404556.52**	2517.47	795.77	437.52	994.70	554.94
case6468rte	**86816.78**	1.02	0.23	0.97	1.52	3.86
case6470rte	**98337.89**	4.92	4.67	7.35	0.27	5.84
case6495rte	**106090.99**	-0.05	-3.25	1.23	0.28	0.80
case6515rte	**109698.87**	0.64	-2.45	-2.49	-0.17	0.56
case9241pegase	**315881.62**	-5.29	1.80	1.38	-6.43	0.77
case13659pegase	**386144.30**	0.19	0.69	1.22	-0.58	0.65

Summary

- problem formulation matters (eliminate spurious constraints etc.)
- clique merging and scaling (often) improve solution accuracy
- MOSEK solves all but one test case to default accuracy
- large test cases can be solved in minutes

Thank you for listening!

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