Application of Polynomial Optimization to Electricity Transmission Networks

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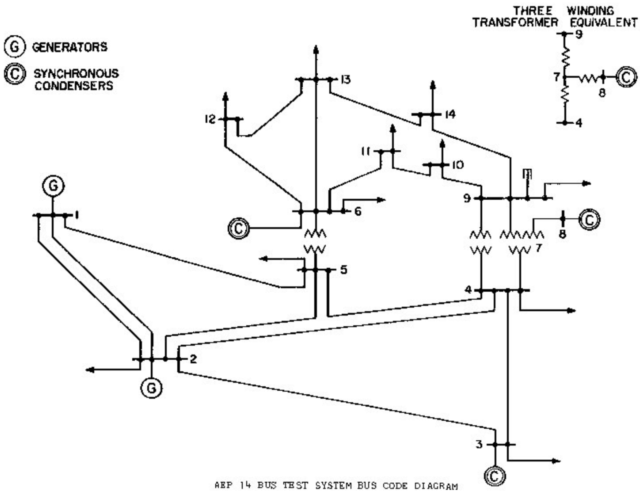
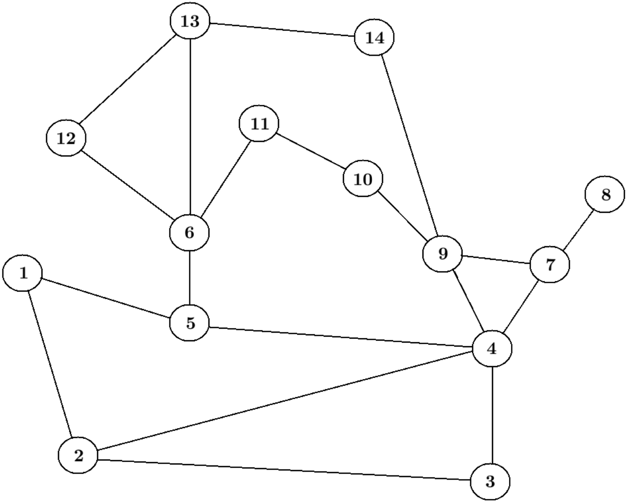
Background and motivations



Underlying graph

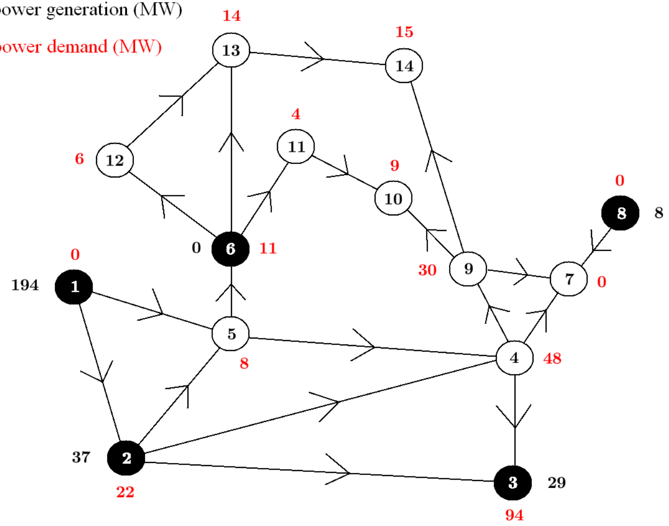


Optimal power flow (1962)

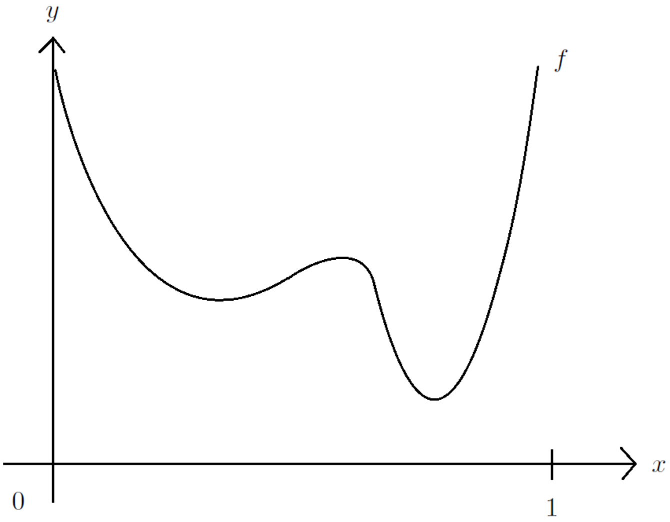
bus linked to a generator

37 active power generation (MW)

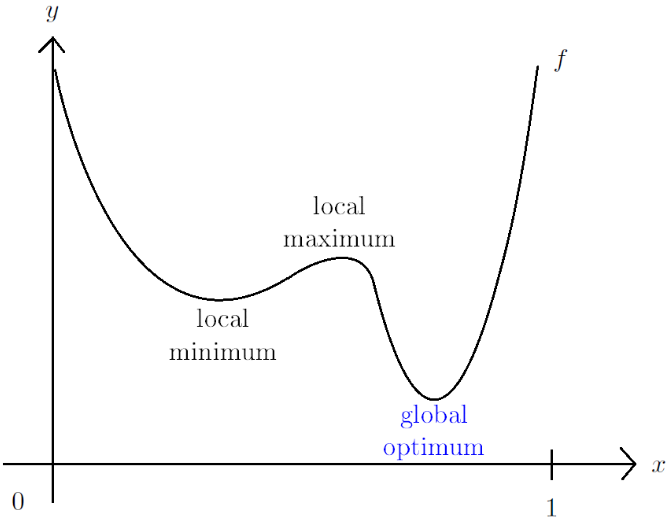
22 active power demand (MW)



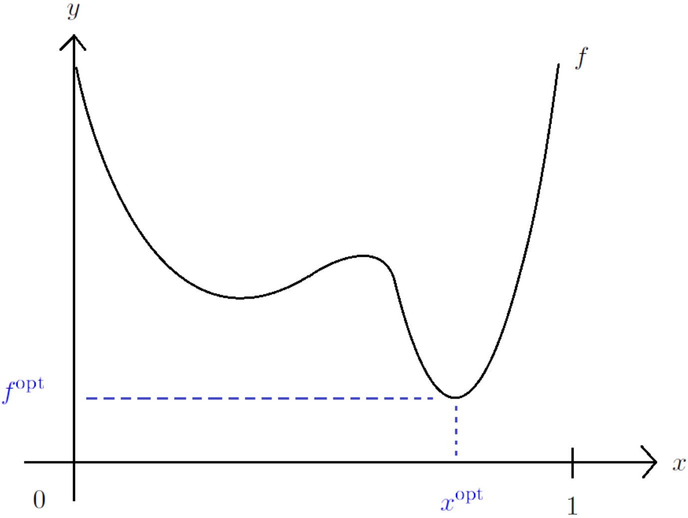
Non-convex optimization



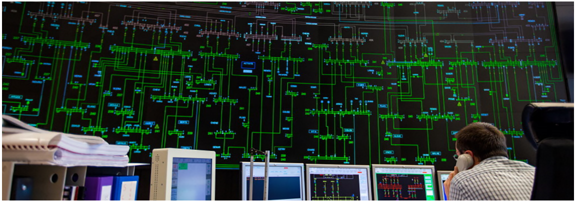
Non-convex optimization



Global value and global solution



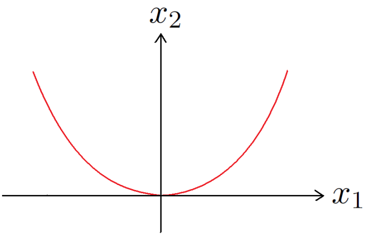
Motivations



Shor relaxation (1987)

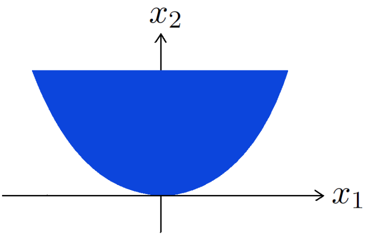
Non-convex problem:

$$\min_{x_1,x_2\in\mathbb{R}} \quad x_1 + 2x_2 \quad \text{s.t.} \quad x_1^2 = x_2$$



Convex relaxation:

$$\min_{x_1,x_2\in\mathbb{R}} \quad x_1 + 2x_2 \quad \text{s.t.} \quad x_1^2 \leqslant x_2$$



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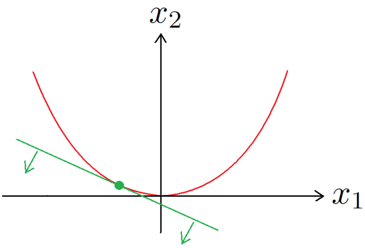
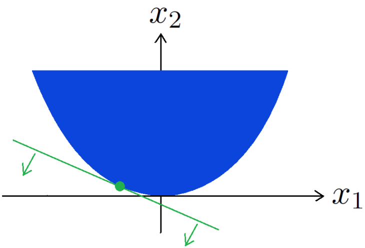
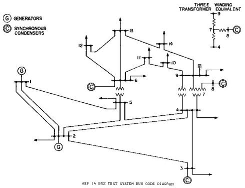


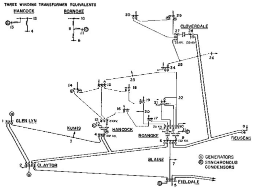
Convex relaxation:

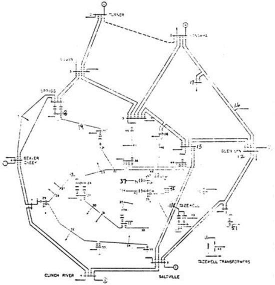
$$\min_{x_1,x_2\in\mathbb{R}} \quad x_1 + 2x_2 \quad \text{s.t.} \quad x_1^2 \leqslant x_2$$

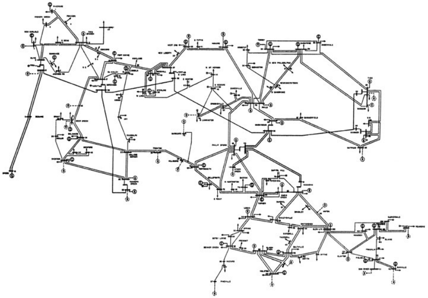


Successful Shor relaxation (Lavaei and Low 2011)

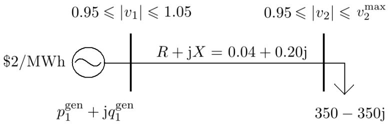


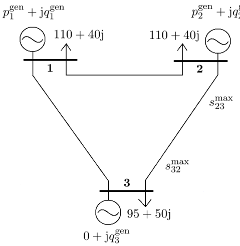


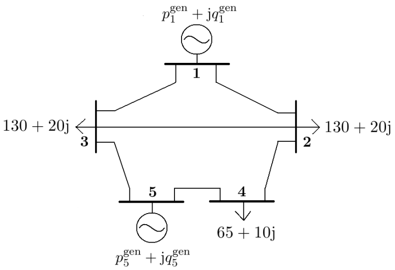


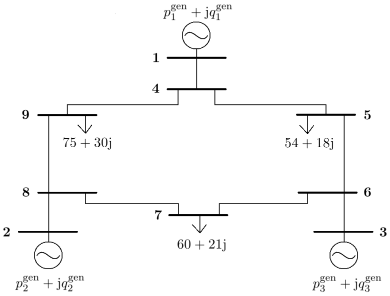


Unsuccessful Shor relaxation (Molzahn *et al.* 2013)

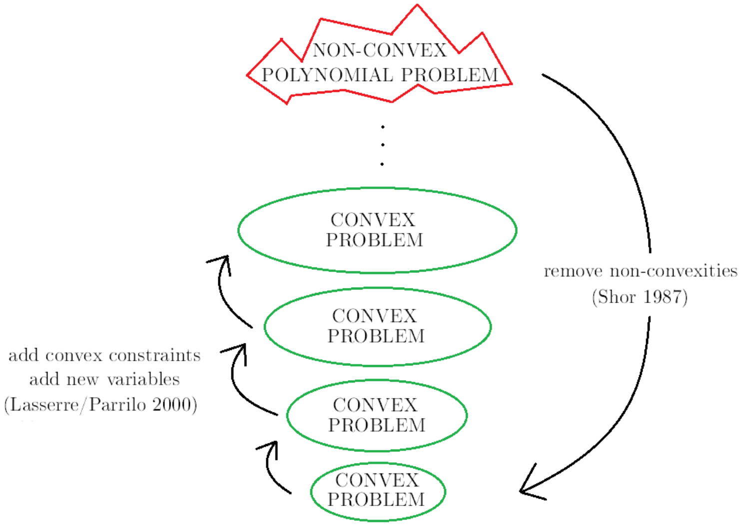




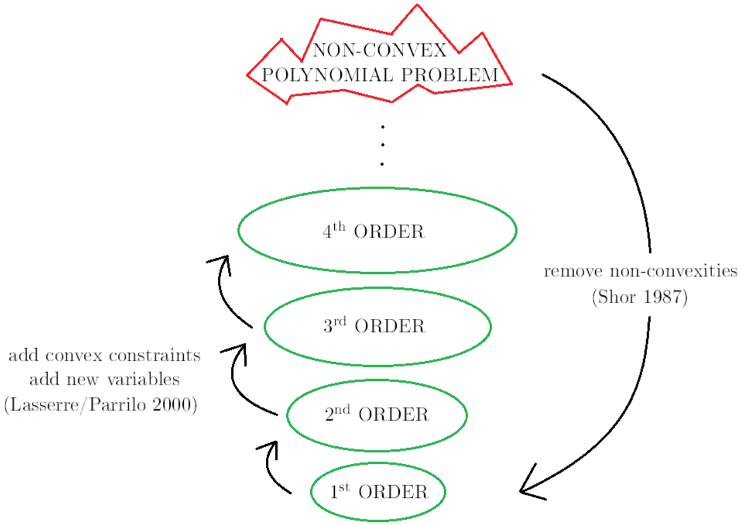




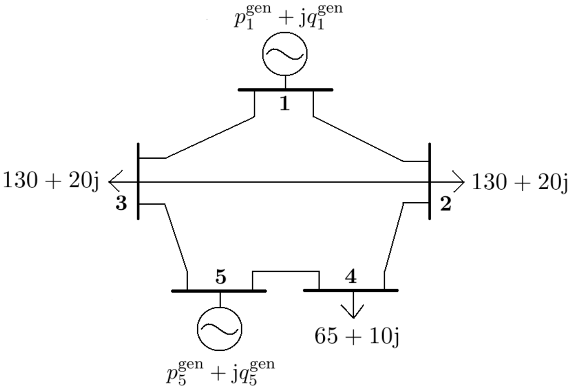
Lasserre hierarchy



Lasserre hierarchy



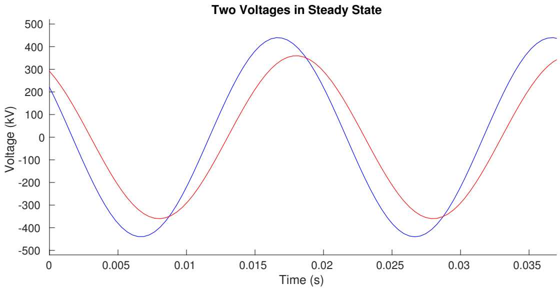
Lasserre hierarchy (J., Maeght, Panciatici, Gilbert 2014)

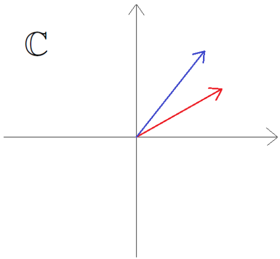


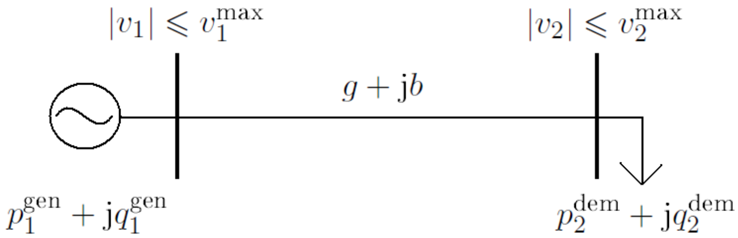
If $q_5^{\text{min}} = -20.51$ MVAR :

- 1st order relaxation yields 954.82 \$/h
- 2nd order relaxation yields 1146.48 \$/h

Alternating current







"g" $=$ conductance

"b" $=$ susceptance

Optimal power flow

Minimize

$$g\ |v_1|^2 - g\ \overline{v}_1 v_2 - g\ \overline{v}_2 v_1 + g\ |v_2|^2$$

over $v_1, v_2 \in \mathbb{C}$ subject to

$$-\frac{g - \mathbf{i}b}{2}\ \overline{v}_1 v_2 - \frac{g + \mathbf{i}b}{2}\ \overline{v}_2 v_1 + g\ |v_2|^2 + p_2^{\text{dem}} = 0$$

$$\frac{b + \mathbf{i}g}{2}\ \overline{v}_1 v_2 + \frac{b - \mathbf{i}g}{2}\ \overline{v}_2 v_1 - b\ |v_2|^2 + q_2^{\text{dem}} = 0$$

$$|v_1|^2 \leqslant (v_1^{\max})^2$$

$$|v_2|^2 \leqslant (v_2^{\max})^2$$

Complex polynomial optimization

Minimize

$$f(z) := \sum_{\alpha,\beta} f_{\alpha,\beta} \bar{z}^{\alpha} z^{\beta} \quad \text{(where } z^{\alpha} := z_1^{\alpha_1} \dots z_n^{\alpha_n})$$

over $z \in \mathbb{C}^n$ subject to

$$g_i(z) := \sum_{\alpha,\beta} g_{i,\alpha,\beta} \bar{z}^{\alpha} z^{\beta} \geqslant 0 \quad , \quad i = 1 \dots m$$

Real polynomial optimization

Minimize

$$f(x) := \sum_{\alpha} f_\alpha x^\alpha \quad (\text{where} \;\; x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n})$$

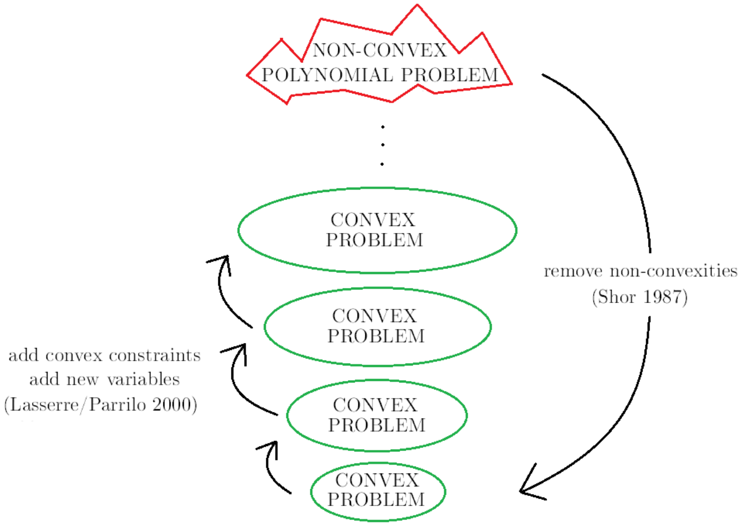
over $x \in \mathbb{R}^n$ subject to

$$g_i(x) := \sum_{\alpha} g_{i,\alpha} x^\alpha \geqslant 0 \quad , \quad i = 1 \dots m$$

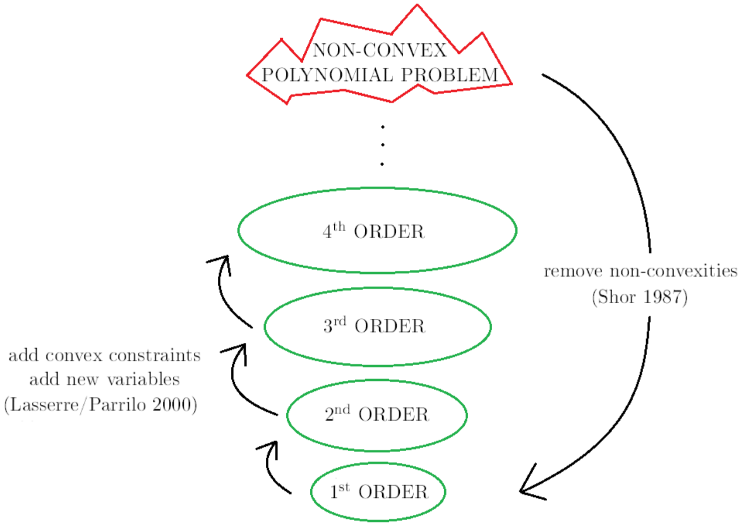
Non-commutative diagram (J. and Molzahn 2015)

$$
\begin{tikzcd}
\text{Non-convex optimization in } \mathbb{C} \arrow[rr, "\text{Identify real and imaginary parts}"] \arrow[d, "\text{Relax}"] & & \text{Non-convex optimization in } \mathbb{R} \arrow[d, "\text{Relax}"'] \\
\text{Convex optimization in } \mathbb{C} \arrow[r, "\text{Identify real and imaginary parts}"] & \text{Convex optimization in } \mathbb{R} \arrow[r, phantom, "\neq"] & \text{Convex optimization in } \mathbb{R}
\end{tikzcd}
$$

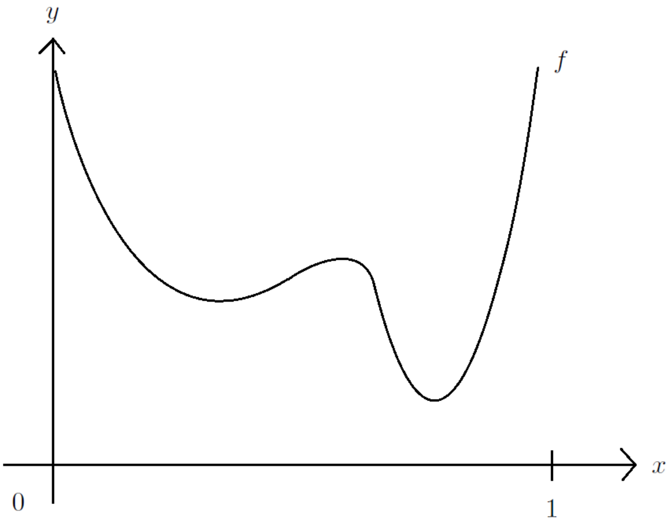
Lasserre hierarchy



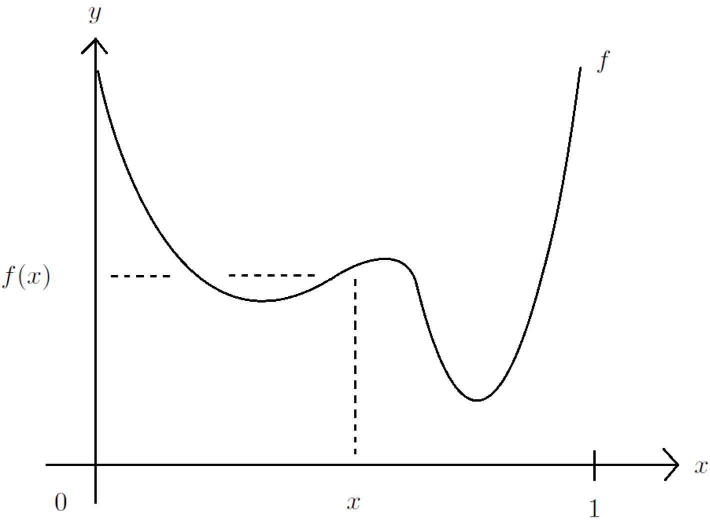
Lasserre hierarchy



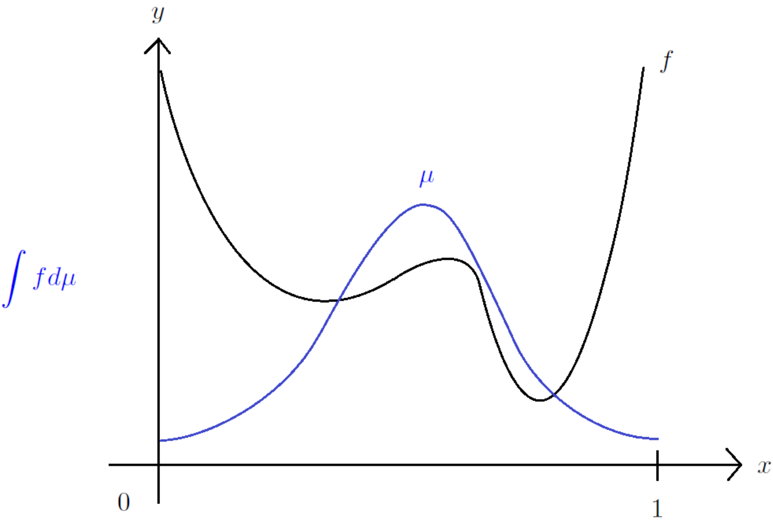
Moment approach (Lasserre 2000) / SOS (Parrilo 2000)



Variable = point

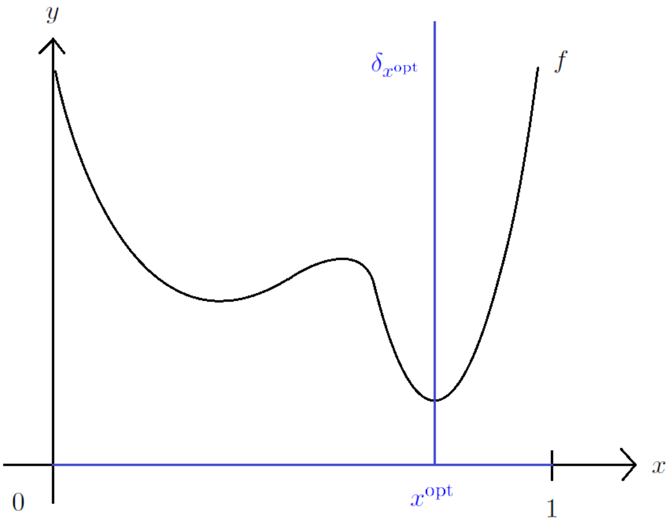


Variable = probability distribution



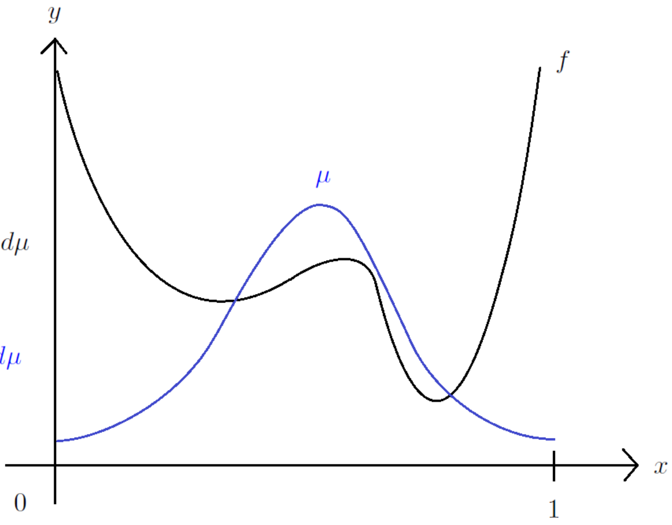
Optimal probability distribution

$$\int f \, d\delta_{x^{\mathrm{opt}}} = f(x^{\mathrm{opt}})$$



Real moment hierarchy (Lasserre 2000)

$$\int f d\mu = \int (\sum_\alpha f_\alpha x^\alpha) d\mu = \sum_\alpha f_\alpha \int x^\alpha d\mu$$



Complex moment hierarchy (J. and Molzahn 2015)

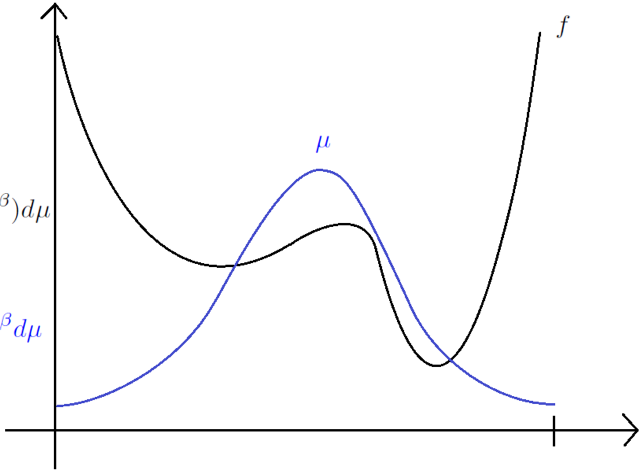
$$\int f \, d\mu$$

$$=$$

$$\int \Big(\sum_{\alpha,\beta} f_{\alpha,\beta} \bar{z}^{\alpha} z^{\beta}\Big) d\mu$$

$$=$$

$$\sum_{\alpha,\beta} f_{\alpha,\beta} \int \bar{z}^{\alpha} z^{\beta} d\mu$$



$$
\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{\begin{array}{c} \sup_{\lambda\in\mathbb{R}} \lambda \\ \text{s.t. } \forall z \in K, \\ f(z) - \lambda \geqslant 0 \end{array}} \arrow[r, "\text{Identify real and}", "\text{imaginary parts}"'] \arrow[dd, "\text{Complex}"', "\text{Hierarchy}"] & \boxed{\begin{array}{c} \sup_{\lambda\in\mathbb{R}} \lambda \\ \text{s.t. } \forall x + \mathbf{i}y \in K, \\ f(x+\mathbf{i}y) - \lambda \geqslant 0 \end{array}} \arrow[d, "\text{Real}"', "\text{Hierarchy}"] \\
 & \boxed{\begin{array}{c} \sup_{\lambda\in\mathbb{R}} \lambda \ \text{ s.t.} \\ \forall x, y \in \mathbb{R}^n, \quad f(x+\mathbf{i}y) - \lambda = \\ \displaystyle\sum_{i=0}^{m} \left(\sum_j \left(\sum_{|\alpha+\beta|\leqslant d} p_{j,\alpha,\beta} x^\alpha y^\beta \right)^2 \right) g_i(x+\mathbf{i}y) \end{array}} \\
\boxed{\begin{array}{c} \sup_{\lambda\in\mathbb{R}} \lambda \ \text{ s.t.} \\ \forall z \in \mathbb{C}^n, \quad f(z) - \lambda = \\ \displaystyle\sum_{i=0}^{m} \left(\sum_j \left| \sum_{|\alpha|\leqslant d} p_{j,\alpha} z^\alpha \right|^2 \right) g_i(z) \end{array}} \arrow[r, "\text{Identify real and}", "\text{imaginary parts}"'] & \boxed{\begin{array}{c} \sup_{\lambda\in\mathbb{R}} \lambda \ \text{ s.t.} \\ \forall x, y \in \mathbb{R}^n, \quad f(x+\mathbf{i}y) - \lambda = \\ \displaystyle\sum_{i=0}^{m} \left(\sum_j \left| \sum_{|\alpha|\leqslant d} p_{j,\alpha} (x+\mathbf{i}y)^\alpha \right|^2 \right) g_i(x+\mathbf{i}y) \end{array}}
\end{tikzcd}
$$

$$\begin{array}{c}\sup_{\lambda\in\mathbb{R}}\lambda\\ \text{s.t. } \forall z\in K,\\ f(z)-\lambda\geqslant 0\end{array} \xrightarrow[\text{imaginary parts}]{\text{Identify real and}} \begin{array}{c}\sup_{\lambda\in\mathbb{R}}\lambda\\ \text{s.t. } \forall x+\mathbf{i}y\in K,\\ f(x+\mathbf{i}y)-\lambda\geqslant 0\end{array}$$

Complex Hierarchy

Real Hierarchy

$$\sup_{\lambda\in\mathbb{R}}\lambda \quad \text{s.t.}$$

$$\forall x,y\in\mathbb{R}^n,\quad f(x+\mathbf{i}y)-\lambda=$$

$$\sum_{i=0}^{m}\left(\sum_j\left|\sum_{|\alpha+\beta|\leqslant d}p_{j,\alpha,\beta}(x-\mathbf{i}y)^{\alpha}(x+\mathbf{i}y)^{\beta}\right|^2\right)g_i(x+\mathbf{i}y)$$

$$\geqslant$$

$$\begin{array}{c}\sup_{\lambda\in\mathbb{R}}\lambda \quad \text{s.t.}\\ \forall z\in\mathbb{C}^n,\quad f(z)-\lambda=\\ \displaystyle\sum_{i=0}^{m}\left(\sum_j\left|\sum_{|\alpha|\leqslant d}p_{j,\alpha}z^{\alpha}\right|^2\right)g_i(z)\end{array} \xrightarrow[\text{imaginary parts}]{\text{Identify real and}} \begin{array}{c}\sup_{\lambda\in\mathbb{R}}\lambda \quad \text{s.t.}\\ \forall x,y\in\mathbb{R}^n,\quad f(x+\mathbf{i}y)-\lambda=\\ \displaystyle\sum_{i=0}^{m}\left(\sum_j\left|\sum_{|\alpha|\leqslant d}p_{j,\alpha}(x+\mathbf{i}y)^{\alpha}\right|^2\right)g_i(x+\mathbf{i}y)\end{array}$$

Example for $z = x + \mathbf{i}y \in \mathbb{C}$

	1	z	z^2	z^3
1	•	•	•	•
$\bar{z}$	•	•	•	•
$\bar{z}^2$	•	•	•	•
$\bar{z}^3$	•	•	•	•

	1	x	y	x^2	xy	y^2	x^3	x^2y	xy^2	y^3
1	•	•	•	•	•	•	•	•	•	•
x	•	•	•	•	•	•	•	•	•	•
y	•	•	•	•	•	•	•	•	•	•
x^2	•	•	•	•	•	•	•	•	•	•
xy	•	•	•	•	•	•	•	•	•	•
y^2	•	•	•	•	•	•	•	•	•	•
x^3	•	•	•	•	•	•	•	•	•	•
x^2y	•	•	•	•	•	•	•	•	•	•
xy^2	•	•	•	•	•	•	•	•	•	•
y^3	•	•	•	•	•	•	•	•	•	•

Optimal power flow (J. and Molzahn 2015)

Case Name	Real hierarchy		Complex hierarchy	
	Val. (MW)	Time (sec)	Val. (MW)	Time (sec)
PL-2383wp	24,990	583.4	24,991	53.9
PL-2736sp	18,334	44.0	18,335	17.8
PL-2737sop	11,397	52.4	11,397	25.7
PL-2746wop	19,210	2,662.4	19,212	124.3
PL-2746wp	25,267	45.9	25,269	18.5
PL-3012wp	27,642	318.7	27,644	141.0
PL-3120sp	21,512	386.6	21,512	193.9
PEGASE-1354	74,043	406.9	74,042	1,132.6
PEGASE-2869	133,944	921.3	133,939	700.8

Quadratically-constrained quadratic program with

$\approx$ 4 000 real variables and $\approx$ 15 000 constraints

Exploiting sparsity (J. and Molzahn 2015)

Complex electric power:

$$v_k \overline{i_k} = v_k \overline{\sum_l i_{kl}} = v_k \overline{\sum_l y_{kl}(v_k - v_l)} = \left(\sum_l \overline{y}_{kl}\right) |v_k|^2 - \sum_l \overline{y}_{kl} v_k \overline{v}_l$$

leads to constraints like

$$2{\color{blue}v_1\overline{v}_1} - (1+j){\color{blue}v_1\overline{v}_2} - (2-j){\color{blue}v_1\overline{v}_3} - (4+3j){\color{blue}v_1\overline{v}_4} = 1 - 3j$$

Monomial sparsity pattern

$$\{(1,2),(1,3),(1,4)\}$$

Constraint sparsity pattern

$$\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$$

Exploiting sparsity (J. and Molzahn 2015)

Until a measure μ can be extracted from y, do:

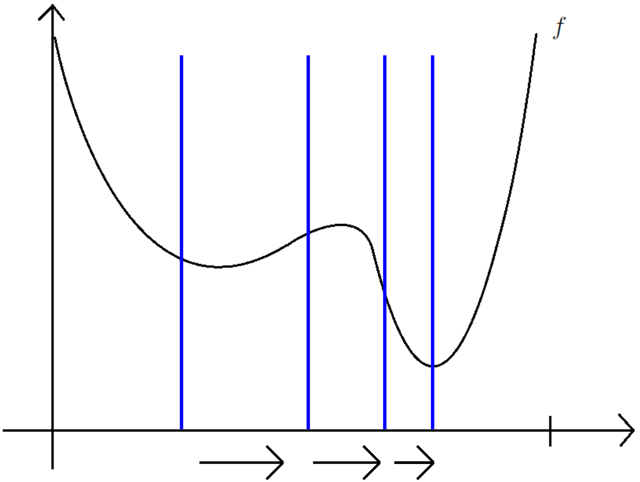
1. Compute a solution y to multi-ordered relaxation $(d_1, \ldots, d_m)$
2. Find a closest measure μ to y:

$$\underset{\mu \text{ Dirac}}{\arg\min} \left\| \left(y_{\alpha,\beta} - \int_{\mathbb{C}^n} \bar{z}^\alpha z^\beta d\mu \right)_{|\alpha|,|\beta|=1} \right\|_{\mathbb{F}}$$

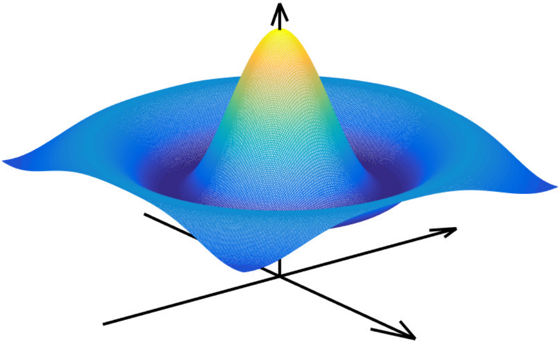
3. Increment $d_i = d_i + 1$ at

$$\underset{1 \leqslant i \leqslant m}{\arg\max} \left| \sum_{\alpha,\beta} g_{i,\alpha,\beta} \left(y_{\alpha,\beta} - \int_{\mathbb{C}^n} \bar{z}^\alpha z^\beta d\mu \right) \right|$$

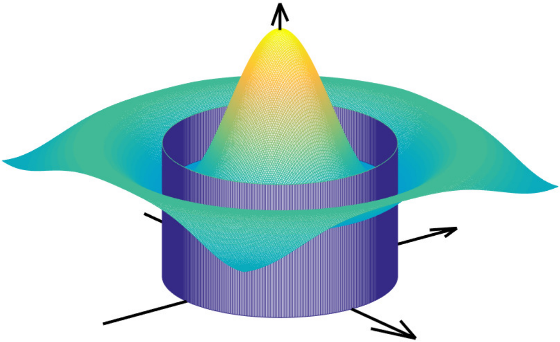
Converging sequence of Dirac measures (J. Molzahn 2015)



Exploiting symmetry (J. and Molzahn 2015)



Exploiting symmetry (J. and Molzahn 2015)



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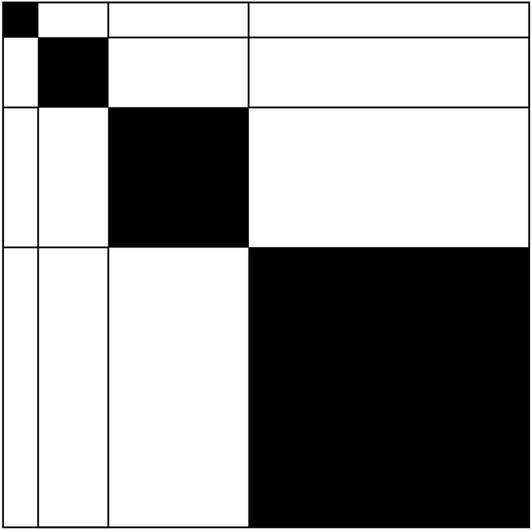
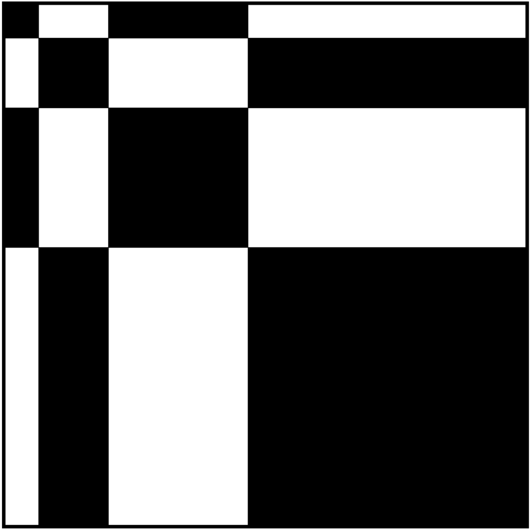
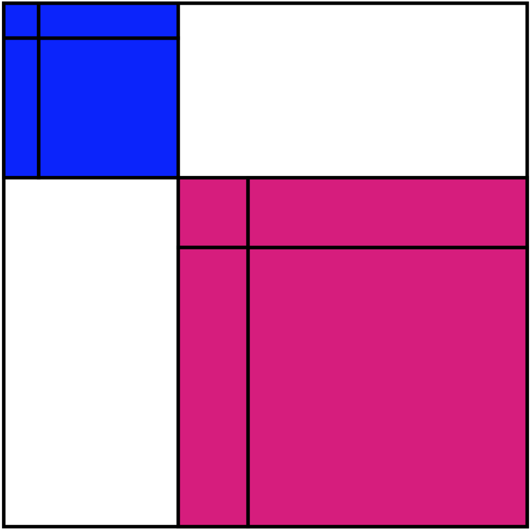


Exploiting symmetry: real version



Exploiting symmetry: real version



Extraction of a global solution

Real version

Given real numbers $(y_\alpha)_{|\alpha|\leqslant 2d}$

$$\exists \mu ? : \quad y_\alpha = \int_K x^\alpha d\mu, \quad \forall |\alpha| \leqslant 2d$$

with $K = \{ \ x \in \mathbb{R}^n \mid g_i(x) \geqslant 0 \ , \quad i = 1, \ldots, m \ \}$.

Complex version

Given complex numbers $(y_{\alpha,\beta})_{|\alpha|,|\beta|\leqslant d}$

$$\exists \mu ? : \quad y_{\alpha,\beta} = \int_K \bar{z}^\alpha z^\beta d\mu, \quad \forall |\alpha|, |\beta| \leqslant d$$

with $K = \{ \ z \in \mathbb{C}^n \mid g_i(z) \geqslant 0 \ , \quad i = 1, \ldots, m \ \}$.

New notion of moment matrix (J. and Molzahn 2015)

Curto and Fialkow (1996):

$$\begin{array}{c|cccccc} & 1 & z & \bar{z} & z^2 & z\bar{z} & \bar{z}^2 \\ \hline 1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bar{z} & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ z & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bar{z}^2 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bar{z}z & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ z^2 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

J. and Molzahn (2015):

$$\begin{array}{c|ccc} & 1 & z & z^2 \\ \hline 1 & \bullet & \bullet & \bullet \\ \bar{z} & \bullet & \bullet & \bullet \\ \bar{z}^2 & \bullet & \bullet & \bullet \end{array}$$

Commutative diagram (J. and Molzahn 2016)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c} \inf_{z\in\mathbb{C}^n} f(z) \\ \text{s.t. } g_i(z) \geqslant 0 \end{array}
\arrow[r, "\text{Hankel property}", "\begin{array}{c} \bar{z}^\alpha z^\beta = \bar{z}^\gamma z^\delta \\ \forall \alpha+\beta=\gamma+\delta \end{array}"']
\arrow[d, "\begin{array}{l} \text{Complex} \\ \text{Hierarchy} \\ y_{\alpha,\beta} = \int_K \bar{z}^\alpha z^\beta d\mu \end{array}"]
&
\begin{array}{c} \inf_{x\in\mathbb{R}^n} f(x) \\ \text{s.t. } g_i(x) \geqslant 0 \end{array}
\arrow[d, "\begin{array}{r} \text{Real} \\ \text{Hierarchy} \\ y_\alpha = \int_K x^\alpha d\mu \end{array}"']
\\
\begin{array}{c} \inf_{y_{\alpha,\beta}} L_y(f) \quad \text{s.t.} \\ M_{d-k_i}(g_i y) \succcurlyeq 0 \\ y_{0,0} = 1 \end{array}
\arrow[r, "\text{Hankel property}", "\begin{array}{c} y_{\alpha,\beta} = y_{\gamma,\delta} \\ \forall \alpha+\beta=\gamma+\delta \end{array}"']
&
\begin{array}{c} \inf_{y_\alpha} L_y(f) \quad \text{s.t.} \\ M_{d-k_i}(g_i y) \succcurlyeq 0 \\ y_0 = 1 \end{array}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{Non-convex optimization in } \mathbb{C} \arrow[rr, "\text{Identify real and}", "\text{imaginary parts}"'] \arrow[d, "\text{Relax}"] & & \text{Non-convex optimization in } \mathbb{R} \arrow[d, "\text{Relax}"'] \\
\text{Convex optimization in } \mathbb{C} \arrow[r, "\text{Identify real and}", "\text{imaginary parts}"'] \arrow[d, "\text{Compute}"] & \text{Convex optimization in } \mathbb{R} \arrow[r, phantom, "\neq"] & \text{Convex optimization in } \mathbb{R} \arrow[d, "\text{Compute}"'] \\
\text{solver in } \mathbb{C} & & \text{solver in } \mathbb{R}
\end{tikzcd}
$$

Solver in complex numbers

Real semidefinite programming:

$$\begin{array}{ll} \inf & \langle C, X\rangle \\ \text{s.t.} & AX = b \in \mathbb{R}^m \\ & X \succcurlyeq 0 \end{array}$$

Complex semidefinite programming (Gilbert and J. 2017):

$$\begin{array}{ll} \inf & \langle C, X\rangle \\ \text{s.t.} & AX = b \in \mathbb{C}^m \\ & X \succcurlyeq 0 \end{array}$$

$$AX = \begin{pmatrix} \langle A_1, X\rangle \\ \vdots \\ \langle A_m, X\rangle \end{pmatrix}, \qquad A_1, \ldots, A_m \in \mathbb{C}^{n\times n}, \qquad \langle U, V\rangle = \operatorname{trace}(U^* V)$$

Complex inequalities

$$\langle A_k, X\rangle \leqslant b_k \quad \text{with} \quad X \succcurlyeq 0$$

$$\Longleftrightarrow$$

$$\begin{cases} \Re(\langle A_k, X\rangle) \leqslant \Re(b_k) \\ \Im(\langle A_k, X\rangle) \leqslant \Im(b_k) \end{cases} \quad \text{with} \quad X \succcurlyeq 0$$

$$\Longleftrightarrow$$

$$\left\langle \begin{pmatrix} A_k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{pmatrix}, \begin{pmatrix} X & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{pmatrix} \right\rangle = b_k \quad \text{and} \quad \begin{pmatrix} X & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{pmatrix} \succcurlyeq 0.$$

complex slack variable $s + it$

Complex lagrangian duality

Primal-dual problems:

$$\begin{array}{lll} \inf & \langle C, X\rangle & \\ \text{s.t.} & AX = b & \times \quad y \in \mathbb{C}^m \\ & X \succcurlyeq 0 & \times \quad S \succcurlyeq 0 \end{array} \qquad \begin{array}{ll} \sup & \Re\langle y, b\rangle \\ \text{s.t.} & \sum_{k=1}^m y_k A_k + \overline{y}_k A_k^* + S = C \\ & S \succcurlyeq 0 \end{array}$$

Necessary and sufficient optimality condition:

$$\begin{cases} AX = b, \quad X \succcurlyeq 0 \\ \sum_{k=1}^m y_k A_k + \overline{y}_k A_k^* + S = C \quad S \succcurlyeq 0 \\ XS = 0 \end{cases}$$

Logarithmic barrier method

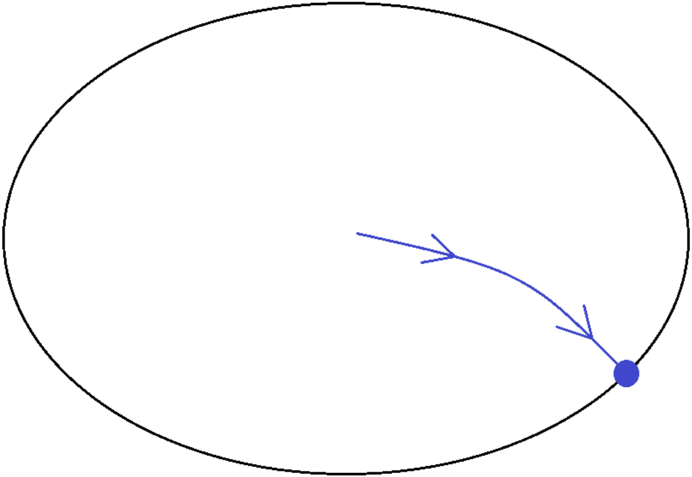
Primal-dual problems:

$$\begin{array}{ll} \inf & \langle C, X\rangle - \mu \log(\det(X)) \\ \text{s.t.} & AX = b \\ & \cancel{X \succcurlyeq 0} \end{array} \qquad \begin{array}{ll} \sup & \Re\langle y, b\rangle + \mu \log(\det(S)) \\ \text{s.t.} & \sum_{k=1}^m y_k A_k + \overline{y}_k A_k^* + S = C \\ & y \in \mathbb{C}^m,\ \cancel{S \succcurlyeq 0} \end{array}$$

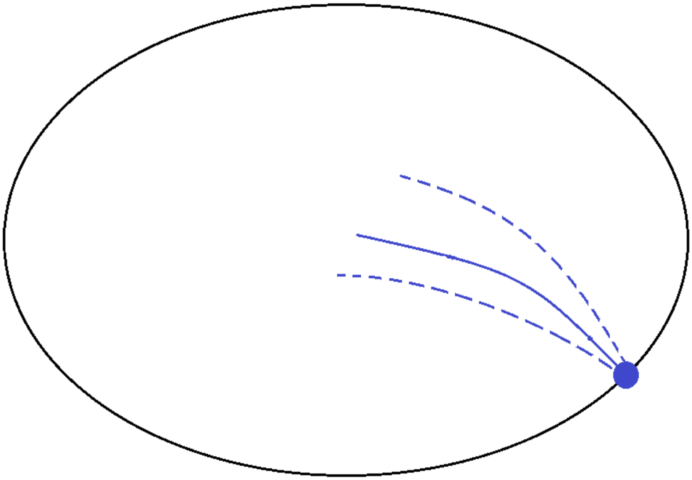
Necessary and sufficient optimality condition:

$$\begin{cases} AX = b, \quad X \succ 0 \\ \sum_{k=1}^m y_k A_k + \overline{y}_k A_k^* + S = C\ , \quad S \succ 0 \\ \cancel{XS = 0} \quad XS = \mu I \end{cases}$$

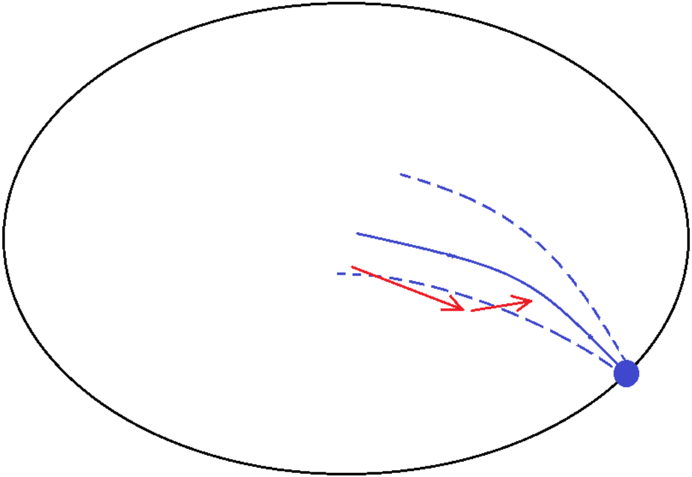
Central path



Neighborhood of central path



Mizuno-Todd-Ye predictor-corrector



Complex Nesterov-Todd direction

Central path with parameter $\mu > 0$:

$$\begin{cases} AX = b, \quad X \succ 0 \\ \sum_{k=1}^{m} y_k A_k + \overline{y}_k A_k^* + S = C \ , \quad S \succ 0 \\ XS = \mu I \end{cases}$$

Complex Nesterov-Todd direction

Newton step $(X + dX, y + dy, S + dS)$:

$$\left\{ \begin{array}{l} A(X + dX) = b \\ \sum_{k=1}^{m} (y_k + dy_k) A_k + \overline{y_k + dy_k} A_k^* + S + dS = C \\ (X + dX)(S + dS) = \mu I \end{array} \right.$$

Primal feasibility

Newton step $(X + dX, y + dy, S + dS)$:

$$\begin{cases} A dX = 0 \\ \sum_{k=1}^{m} (y_k + dy_k) A_k + \overline{y_k + dy_k} A_k^* + S + dS = C \\ (X + dX)(S + dS) = \mu I \end{cases}$$

Dual feasibility

Newton step $(X + dX, y + dy, S + dS)$:

$$\begin{cases} A dX = 0 \\ \sum_{k=1}^{m} dy_k A_k + \overline{dy_k} A_k^* + dS = 0 \\ (X + dX)(S + dS) = \mu I \end{cases}$$

Complementary constraint

Newton step $(X + dX, y + dy, S + dS)$:

$$\begin{cases} AdX = 0 \\ \sum_{k=1}^{m} dy_k A_k + \overline{dy_k} A_k^* + dS = 0 \\ XS + X\,dS + dX\,S + dX\,dS = \mu I \end{cases}$$

Discard high-order term

Newton step $(X + dX, y + dy, S + dS)$:

$$\left\{ \begin{array}{l} A dX = 0 \\ \sum_{k=1}^{m} dy_k A_k + \overline{dy_k} A_k^* + dS = 0 \\ XS + X dS + dX S + \cancel{dX\, dS} = \mu I \end{array} \right.$$

Symmetrization of complementary constraint

Newton step $(X + dX, y + dy, S + dS)$:

$$\begin{cases} A dX = 0 \\ \sum_{k=1}^{m} dy_k A_k + \overline{dy_k} A_k^* + dS = 0 \\ (XS + X dS + dX S) + (XS + X dS + dX S)^* = 2\mu I \end{cases}$$

Search for symmetric solution

Newton step $(X + dX, y + dy, S + dS)$:

$$\begin{cases} A dX = 0 \\ \sum_{k=1}^{m} dy_k A_k + \overline{dy_k} A_k^* + dS = 0 \\ X dS + S dX + dS X + dX S = 2\mu I - XS - SX \end{cases}$$

What if primal-dual variables are equal $X = S$?

Newton step $(X + dX, y + dy, S + dS)$:

$$\begin{cases} AdX = 0 \\ \sum_{k=1}^{m} dy_k A_k + \overline{dy_k} A_k^* + dS = 0 \\ XdS + SdX + dSX + dXS = 2\mu I - XS - SX \end{cases}$$

What if primal-dual variables are equal $V := X = S$?

Newton step $(X + dX, y + dy, S + dS)$:

$$\begin{cases} A dX = 0 \\ \sum_{k=1}^{m} dy_k A_k + \overline{dy_k} A_k^* + dS = 0 \\ V dS + V dX + dS\, V + dX\, V = 2\mu I - 2V^2 \end{cases}$$

Lyapunov equation

Newton step $(X + dX, y + dy, S + dS)$:

$$\begin{cases} A dX = 0 \\ \sum_{k=1}^{m} dy_k A_k + \overline{dy_k} A_k^* + dS = 0 \\ V(dX + dS) + (dX + dS)V + 2V^2 - 2\mu I = 0 \end{cases}$$

Lyapunov equation

Newton step $(X + dX, y + dy, S + dS)$:

$$\begin{cases} A dX = 0 \\ \sum_{k=1}^{m} dy_k A_k + \overline{dy_k} A_k^* + dS = 0 \\ V(dX + dS) + (dX + dS)V + 2V^2 - 2\mu I = 0 \end{cases}$$

whose unique solution is

$$dX + dS = \mu V^{-1} - V$$

How to make primal-dual variables equal $X = S$?

1. For all invertible matrix D:

$$XS = \mu I \quad \iff \quad (D^{-1}XD^{-1})(DSD) = \mu I$$

2. We'd like to have $D^{-1}XD^{-1} = DSD$

3. This is possible!

$$D = (S^{-1/2}(S^{1/2}XS^{1/2})^{1/2}S^{-1/2})^{1/2}$$

Plug in the Lyapunov solution

Newton step $(X + dX, y + dy, S + dS)$:

$$\begin{cases} A dX = 0 \\ \sum_{k=1}^{m} dy_k A_k + \overline{dy_k} A_k^* + dS = 0 \\ D^{-1} dX D^{-1} + D dS D = \mu (DSD)^{-1} - D^{-1} X D^{-1} \end{cases}$$

Multiply on left and right sides by D

Newton step $(X + dX, y + dy, S + dS)$:

$$\begin{cases} A dX = 0 \\ \sum_{k=1}^{m} dy_k A_k + \overline{dy_k} A_k^* + dS = 0 \\ dX + D^2 dS D^2 = \mu S^{-1} - X \end{cases}$$

Multiply on left side by A

Newton step $(X + dX, y + dy, S + dS)$:

$$\begin{cases} A dX = 0 \\ \sum_{k=1}^{m} dy_k A_k + \overline{dy_k} A_k^* + dS = 0 \\ A D^2 dS D^2 = \mu A S^{-1} - b \end{cases}$$

Substitution of dual variables

Newton step $(X + dX, y + dy, S + dS)$:

$$\begin{cases} AdX = 0 \\ \sum_{k=1}^m dy_k A_k + \overline{dy_k} A_k^* + dS = 0 \\ AD^2(\sum_{k=1}^m dy_k A_k + \overline{dy_k} A_k^*) D^2 = b - \mu A S^{-1} \end{cases}$$

Expansion

Newton step $(X + dX, y + dy, S + dS)$:

$$\left\{ \begin{array}{l} AdX = 0 \\ \sum_{k=1}^{m} dy_k A_k + \overline{dy_k} A_k^* + dS = 0 \\ \sum_{k=1}^{m} dy_k AD^2 A_k D^2 + \overline{dy_k} AD^2 A_k^* D^2 = b - \mu AS^{-1} \end{array} \right.$$

Linear algebra

Given $M, N \in \mathbb{C}^{m\times m}$ and $p \in \mathbb{C}^m$, the system

$$Mz + N\bar{z} = p \; , \qquad z \in \mathbb{C}^m$$

can be solved by identifying real and imaginary parts

$$\begin{pmatrix} \Re(M+N) & -\Im(M-N) \\ \Im(M+N) & \Re(M-N) \end{pmatrix} \begin{pmatrix} \Re z \\ \Im z \end{pmatrix} = \begin{pmatrix} \Re p \\ \Im p \end{pmatrix}$$

Proposition (Josz 2016)

Given $M, N \in \mathbb{C}^{m\times m}$ and $p \in \mathbb{C}^m$, the system

$$Mz + N\bar{z} = p \ , \qquad z \in \mathbb{C}^m$$

has a unique solution if and only if

$$\begin{pmatrix} \overline{M} & \overline{N} \\ N & M \end{pmatrix} \text{ is invertible}$$

in which case it may be reduced to

$$(M - N\overline{M}^{-1}\overline{N})z = p - N\overline{M}^{-1}\overline{p}$$

$$\boxed{\inf \ \langle C, X\rangle \quad \text{s.t.} \quad \langle A_i, X\rangle = b_i \ , \ i = 1, \ldots, m \ , \ X \succcurlyeq 0}$$

$$\begin{pmatrix} \overline{M} & \overline{N} \\ N & M \end{pmatrix} \text{ is the Gram matrix of } A_1, \ldots, A_m, A_1^*, \ldots A_m^*$$

Proposition (Josz 2016)

Given $M, N \in \mathbb{C}^{m\times m}$ and $p \in \mathbb{C}^m$, the system

$$Mz + N\bar{z} = p \ , \qquad z \in \mathbb{C}^m$$

has a unique solution if and only if

$$\begin{pmatrix} \overline{M} & \overline{N} \\ N & M \end{pmatrix} \text{ is invertible}$$

in which case it may be reduced to

$$(M - N\overline{M}^{-1}\overline{N})z = p - N\overline{M}^{-1}\overline{p}$$

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$\begin{pmatrix} \overline{M} & \overline{N} \\ N & M \end{pmatrix}$ is the Gram matrix of $A_1, \ldots, A_m, A_1^*, \ldots A_m^*$

Conclusion and perspectives

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\begin{array}{c}\text{Non-convex}\\ \text{optimization}\\ \text{in } \mathbb{C}\end{array} \arrow[rr, "\text{Identify real and}", "\text{imaginary parts}"'] \arrow[d, "\text{Relax}"] & & \begin{array}{c}\text{Non-convex}\\ \text{optimization}\\ \text{in } \mathbb{R}\end{array} \arrow[d, "\text{Relax}"'] \\
\begin{array}{c}\text{Convex}\\ \text{optimization}\\ \text{in } \mathbb{C}\end{array} \arrow[r, "\text{Identify real and}", "\text{imaginary parts}"'] \arrow[d, "\text{Compute}"] & \begin{array}{c}\text{Convex}\\ \text{optimization}\\ \text{in } \mathbb{R}\end{array} \arrow[r, phantom, "\neq"] & \begin{array}{c}\text{Convex}\\ \text{optimization}\\ \text{in } \mathbb{R}\end{array} \arrow[d, "\text{Compute}"'] \\
\text{solver in } \mathbb{C} & & \text{solver in } \mathbb{R}
\end{tikzcd}
$$

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Thank you for your attention!

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