Conic optimization
Convex cones
Semidefinite matrices
Linear cone problems

Conic modeling
Simple cones
Nearest correlation
Linear matrix inequalities
Eigenvalue optimization
Combinatorial relaxations
Sum-of-squares relaxations
Nonnegative polynomials

Conclusions

Semidefinite optimization

with MOSEK

Joachim Dahl

MOSEK ApS

INFORMS annual meeting
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- Convex cones
- Semidefinite matrices
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What is a convex cone?

S is a *convex cone* if

$$x \in S \quad \Longleftrightarrow \quad \alpha \cdot x \in S, \; \forall \alpha \geq 0$$

Simple examples

- Nonnegative orthant, $x \geq 0$.
- Quadratic cone,

$$\mathcal{Q}^n = \left\{ x \in \mathbb{R}^n \mid x_1 \geq \sqrt{x_2^2 + \ldots + x_n^2} \right\}.$$

also known as *second-order* or *Lorentz* cone.

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Semidefinite matrices

A symmetric matrix $X \in \mathcal{S}^n$ is positive semidefinite iff

- all eigenvalues are nonnegative.
- it can be factored as $X = VV^T$.
- $z^T X z \geq 0, \; \forall z \in \mathbb{R}^n$.

Cone of semidefinite matrices

$$\mathcal{S}^n_+ = \left\{ X \in \mathcal{S}^n \mid z^T X z \geq 0, \; \forall z \in \mathbb{R}^n \right\}.$$

Notation: $X \succeq Y \iff (X - Y) \in \mathcal{S}^n_+$.

Matrix inner-products and norms

$$\langle A, B \rangle := \operatorname{trace}(A^T B) = \sum_{ij} A_{ij} B_{ij}$$

$$\|A\|_F^2 := \langle A, A \rangle = \sum_{ij} A_{ij}^2.$$

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```
>> A=[2,1,1; 1,2,1; 1,1,1]

A =

     2     1     1
     1     2     1
     1     1     1

>> [U, D]=eig(A) % A = U*D*U'

U =

    0.3251    0.7071    0.6280
    0.3251   -0.7071    0.6280
   -0.8881         0    0.4597


D =

    0.2679         0         0
         0    1.0000         0
         0         0    3.7321
```

```
>> V=U*sqrt(D)

V =

    0.1683    0.7071    1.2131
    0.1683   -0.7071    1.2131
   -0.4597         0    0.8881

>> V*V'           % V is a factor of A

ans =

    2.0000    1.0000    1.0000
    1.0000    2.0000    1.0000
    1.0000    1.0000    1.0000

>> A(:)'*A(:)   % squared Frobenius norm

ans =

    15.0000

>> sum(diag(D).^2)

ans =

    15.0000
```

Linear cone problems

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \in \mathcal{C} \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & c - A^T y = s \\ & s \in \mathcal{C} \end{array}$$

where $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_p$ is a product of cones.

Admissible cones

- Nonnegative orthant $x \geq 0$.
- Quadratic cone $\mathcal{Q}^n$.
- Rotated quadratic cone,

$$\mathcal{Q}_r^n = \left\{ x \in \mathbb{R}^n \mid 2x_1 x_2 \geq x_3^2 + \ldots + x_n^2,\ x_1, x_2 \geq 0 \right\}.$$

- Semidefinite cone $\mathcal{S}_+^n$.

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- Semidefinite cone $\mathcal{S}_+^n$.

Example problem:

$$
\begin{array}{lll}
\text{minimize} & \left\langle \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, X \right\rangle + z_1 & \\
\text{subject to} & \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, X \right\rangle + z_1 & = \quad 1 \\
 & \left\langle \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, X \right\rangle + z_2 + z_3 & = \quad 1/2 \\
 & (z_1, z_2, z_3) \in Q^3, \; X \in \mathcal{S}_+^3 &
\end{array}
$$

A standard linear cone problem with

$$x = \begin{pmatrix} z_1 & z_2 & z_3 & X_{11} & X_{21} & X_{31} & X_{12} & X_{22} & X_{23} & X_{13} & X_{23} & X_{33} \end{pmatrix}^T$$

$$c = \begin{pmatrix} 1 & 0 & 0 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 2 \end{pmatrix}^T$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 1 & 1/2 \end{pmatrix}^T$$

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$$

$$
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$$

```
>> [x(1:3), s(1:3)]

ans =

    0.2544    0.4552
    0.1799   -0.3219
    0.1799   -0.3219

>> X

X =

    0.2173   -0.2600    0.2173
   -0.2600    0.3111   -0.2600
    0.2173   -0.2600    0.2173

>> S

S =

    1.1333    0.6781   -0.3219
    0.6781    1.1333    0.6781
   -0.3219    0.6781    1.1333

>> c'*x - b'*y

ans =

    4.3340e-08

>> norm(A*x-b)

ans =

   2.4379e-08

>> norm(c-A'*y-s)

ans =

   1.6585e-08

>> x'*s

ans =

   2.9971e-08

>> [eig(X), eig(S)]

ans =

    0.0000   -0.0000
    0.0000    1.4552
    0.7456    1.9448

>> x(1)-norm(x(2:3))

ans =

   8.0901e-09

>> s(1)-norm(s(2:3))

ans =

   2.2202e-08
```

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Simple examples of quadratic cones

- Absolute values

$$|x| \leq t \quad \Longleftrightarrow \quad (t, x) \in \mathcal{Q}^2$$

- Euclidean norms

$$\|x\| \leq t \quad \Longleftrightarrow \quad (t, x) \in \mathcal{Q}^{n+1}$$

- Squared euclidean norms

$$\|x\|^2 \leq t \quad \Longleftrightarrow \quad (t, 1/2, x) \in \mathcal{Q}_r^{n+2}$$

- Hyperbolic sets

$$\frac{1}{x} \leq t, \, x > 0 \quad \Longleftrightarrow \quad (t, x, \sqrt{2}) \in \mathcal{Q}_r^3$$

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Very simple examples of semidefinite cones

- Nonnegativity

$$x \geq 0 \quad \Longleftrightarrow \quad \mathrm{diag}(x) \succeq 0.$$

- Quadratic cones

$$\begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix} \succeq 0 \quad \Longleftrightarrow \quad x_1 x_2 \geq x_3^2, \quad x_1, x_2 \geq 0,$$

in other words,

$$\begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix} \succeq 0 \quad \Longleftrightarrow \quad (x_1, x_2, x_3/\sqrt{2}) \in \mathcal{Q}_r^3.$$

A similar result for n-dimensional quadratic cones.

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A picture is worth a thousand words...

The pillow

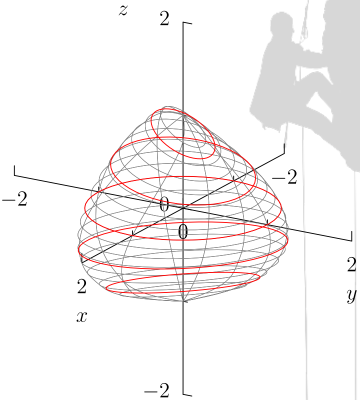
Consider the set:

$$\begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0.$$

- Exterior is a *spectrahedron*.
- Can be characterized as

$$x^2 + y^2 + z^2 - 2xyx = 1.$$

- Ellipsoids for fixed $z \in [-1, 1]$.



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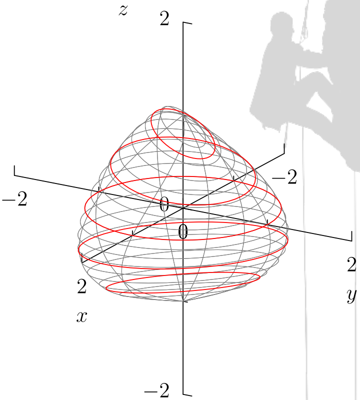

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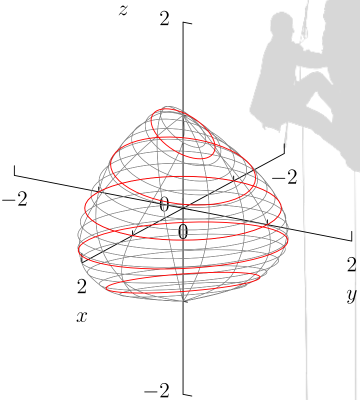

Nearest correlation problem

Consider the set

$$S = \{X \in \mathcal{S}^n_+ \mid q_{ii} = 1,\ i = 1, \dots, n\}.$$

For $A \in \mathcal{S}^n$ the *nearest correlation matrix* is

$$X^\star = \arg\min_{X \in S} \|A - X\|_F.$$

A conic formulation

$$\begin{array}{lrcl} \text{minimize} & t & & \\ \text{subject to} & \|\text{vec}(A - X)\| & \leq & t \\ & \text{diag}(X) & = & e \\ & X & \succeq & 0 \end{array}$$

where $\text{vec}(X) = (x_{11}, x_{21}, \dots, x_{n1}, x_{12}, \dots x_{nn})$.

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Linear matrix functions

Consider a matrix-valued function $F : \mathbb{R}^m \mapsto \mathcal{S}^n$,

$$F(x) = F_0 + x_1 F_1 + \cdots + x_m F_m$$

where $F_i \in \mathcal{S}^n$.

- The inequality

$$F_0 + x_1 F_1 + \cdots + x_m F_m \succeq 0$$

  is called a *linear matrix inequality* (LMI).
- Corresponds to conic dual constraints,

$$C - (y_1 A_1 + \cdots + y_m A_m) = S, \quad S \succeq 0.$$

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Eigenvalue optimization

$$F(x) = F_0 + x_1 F_1 + \cdots + x_m F_m, \quad F_i \in \mathcal{S}_m.$$

- Minimize largest eigenvalue $\lambda_1(F(x))$:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & \gamma I \succeq F(x), \end{array}$$

- Maximize smallest eigenvalue $\lambda_n(F(x))$:

$$\begin{array}{ll} \text{maximize} & \gamma \\ \text{subject to} & F(x) \succeq \gamma I, \end{array}$$

- Minimize eigenvalue spread $\lambda_1(F(x)) - \lambda_n(F(x))$:

$$\begin{array}{ll} \text{minimize} & \gamma - \lambda \\ \text{subject to} & \gamma I \succeq F(x) \succeq \lambda I, \end{array}$$

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Eigenvalue optimization

$$F(x) = F_0 + x_1 F_1 + \cdots + x_m F_m, \quad F_i \in \mathcal{S}_m.$$

- Minimize largest eigenvalue $\lambda_1(F(x))$:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & \gamma I \succeq F(x), \end{array}$$

- Maximize smallest eigenvalue $\lambda_n(F(x))$:

$$\begin{array}{ll} \text{maximize} & \gamma \\ \text{subject to} & F(x) \succeq \gamma I, \end{array}$$

- Minimize eigenvalue spread $\lambda_1(F(x)) - \lambda_n(F(x))$:

$$\begin{array}{ll} \text{minimize} & \gamma - \lambda \\ \text{subject to} & \gamma I \succeq F(x) \succeq \lambda I, \end{array}$$

Minimizing matrix norms

$$F(x) = F_0 + x_1 F_1 + \cdots + x_m F_m, \quad F_i \in \mathbb{R}^{n \times p}.$$

- (Standard) matrix norm: $\|F(x)\|_2 = \max_k \sigma_k(F(x))$,

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} tI & F(x)^T \\ F(x) & tI \end{bmatrix} \succeq 0, \end{array}$$

- Nuclear norm: $\|F(x)\|_* = \sum_k \sigma_k(F(x))$,

$$\begin{array}{ll} \text{minimize} & \text{trace}(U+V)/2 \\ \text{subject to} & \begin{bmatrix} U & F(x)^T \\ F(x) & V \end{bmatrix} \succeq 0. \end{array}$$

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A binary quadratic problem

Binary quadratic problem

We consider a binary problem.

$$\begin{array}{ll} \text{minimize} & x^T Q x + c^T x \\ \text{subject to} & x_i \in \{0,1\}, \quad i = 1,\ldots,n. \end{array}$$

where Q can be indefinite.

- Very difficult non-convex problem.
- In general we have to explore 2^n different objectives.
- Instead use a *semidefinite relaxation*.

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Lifting of binary constraints

Rewrite binary constraints $x_i \in \{0, 1\}$:

$$x_i^2 = x_i \quad \Longleftrightarrow \quad X = xx^T, \quad \mathrm{diag}(X) = x.$$

Still non-convex, since $\mathrm{rank}(X) = 1$.

Semidefinite relaxation of binary constraints

$$X \succeq xx^T, \quad \mathrm{diag}(X) = x.$$

Note that:

$$X \succeq xx^T \quad \Longleftrightarrow \quad \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0,$$

which is a linear matrix inequality.

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The lifted non-convex problem

$$\begin{array}{ll} \text{minimize} & \langle Q, X\rangle + c^T x \\ \text{subject to} & \operatorname{diag}(X) = x \\ & X = xx^T \end{array}$$

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- Relaxation is exact if $X = xx^T$.
- Otherwise can be strengthened, e.g., by adding $X_{ij} \geq 0$.
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Relaxations for boolean optimization

Same approach used for boolean constraints $x_i \in \{-1, +1\}$.

Lifting of boolean constraints

Rewrite boolean constraints $x_i \in \{-1, 1\}$:

$$x_i^2 = 1 \quad \Longleftrightarrow \quad X = xx^T, \quad \mathrm{diag}(X) = e.$$

Semidefinite relaxation of boolean constraints

$$X \succeq xx^T, \quad \mathrm{diag}(X) = e.$$

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Sum-of-squares relaxations

- f: multivariate polynomial of degree $2d$.
- $v_d = (1, x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^2, \ldots, x_n^d)$.
  Vector of monomials of degree d or less.

Sum-of-squares representation

f is a sum-of-squares (SOS) iff

$$f(x_1, \ldots, x_n) = v_d^T Q v_d, \quad Q \succeq 0.$$

If $X = LL^T$ then

$$f(x_1, \ldots, x_n) = v_d^T LL^T v_d = \sum_{i=1}^{m} (l_i^T v_d)^2.$$

Is obviously **sufficient** for $f(x_1, \ldots, x_n) \geq 0$.

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A simple example

Consider
$$f(x,z) = 2x^4 + 2x^3z - x^2z^2 + 5z^4,$$
homogeneous of degree 4, so we only need
$$v = \begin{pmatrix} x^2 & xz & z^2 \end{pmatrix}.$$

Comparing cofficients of $f(x,z)$ and $v^TQv = \langle Q, vv^T\rangle$,
$$\langle Q, vv^T\rangle = \langle \begin{pmatrix} q_{00} & q_{01} & q_{02} \\ q_{10} & q_{11} & q_{12} \\ q_{20} & q_{21} & q_{22} \end{pmatrix}, \begin{pmatrix} x^4 & x^3z & x^2z^2 \\ x^3z & x^2z^2 & xz^3 \\ x^2z^2 & xz^3 & z^4 \end{pmatrix} \rangle$$
we see that $f(x,z)$ is SOS iff $Q \succeq 0$ and
$$q_{00} = 2, \quad 2q_{10} = 2, \quad 2q_{20} + q_{11} = -1, \quad 2q_{21} = 0, \quad q_{22} = 5.$$

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Applications in global optimization

$$f(x,z) = 4x^2 - \frac{21}{10}x^4 + \frac{1}{3}x^6 + xz - 4z^2 + 4z^4$$

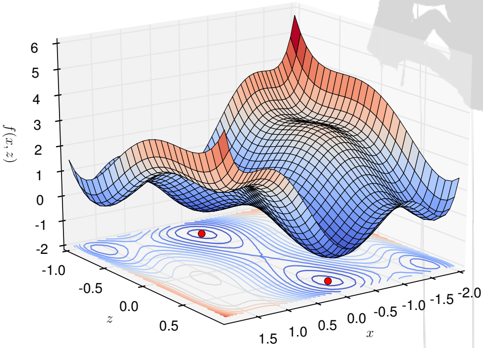
Global lower bound

Replace non-tractable problem,

$$\text{minimize} \quad f(x,z)$$

by a tractable lower bound

$$\begin{array}{ll} \text{maximize} & t \\ \text{subject to} & f(x,z) - t \text{ is SOS.} \end{array}$$



Relaxation finds the global optimum $t = -1.031$.

Essentially due to Shor, 1987.

$$f(x,z) - t = 4x^2 - \frac{21}{10}x^4 + \frac{1}{3}x^6 + xz - 4z^2 + 4z^4 - t$$

$$vv^T = \begin{pmatrix}
1 & x & z & x^2 & xz & z^2 & x^3 & x^2z & xz^2 & z^3 \\
x & x^2 & xz & x^3 & x^2z & xz^2 & x^4 & x^3z & x^2z^2 & xz^3 \\
z & xz & z^2 & x^2z & xz^2 & z^3 & x^3z & x^2z^2 & xz^3 & z^4 \\
x^2 & x^3 & x^2z & x^4 & x^3z & x^2z^2 & x^5 & x^4z & x^3z^2 & x^2z^3 \\
xz & x^2z & xz^2 & x^3z & x^2z^2 & xz^3 & x^4z & x^3z^2 & x^2z^3 & xz^4 \\
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x^3 & x^4 & x^3z & x^5 & x^4z & x^3z^2 & x^6 & x^5z & x^4z^2 & x^3z^3 \\
x^2z & x^3z & x^2z^2 & x^4z & x^3z^2 & x^2z^3 & x^5z & x^4z^2 & x^3z^3 & x^2z^4 \\
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z^3 & xz^3 & z^4 & x^2z^3 & xz^4 & z^5 & x^3z^3 & x^2z^4 & xz^5 & z^6
\end{pmatrix}$$

By comparing cofficients of v^TQv and $f(x,z) - t$:

$$q_{00} = -t, \quad (2q_{30} + q_{11}) = 4, \quad (2q_{72} + q_{44}) = -\frac{21}{10}, \quad q_{77} = \frac{1}{3}$$

$$2(q_{51} + q_{32}) = 1, \quad (2q_{61} + q_{33}) = -4, \quad (2q_{10,3} + q_{66}) = 4$$

$$2q_{10} = 0, \quad 2q_{20} = 0, \quad 2(q_{71} + q_{42}) = 0, \quad \ldots$$

A standard SDP with a 10×10 variable and 27 constraints.

Nonnegative polynomials

- Univariate polynomial of degree $2n$:

$$f(x) = c_0 + c_1 x + \cdots + c_{2n} x^{2n}.$$

- Nonnegativity is **equivalent** to SOS, i.e.,

$$f(x) \geq 0 \quad \Longleftrightarrow \quad f(x) = v^T Q v, \quad Q \succeq 0$$

with $v = (1, x, \ldots, x^n)$.

- Simple extensions for nonnegativity on a subinterval $I \subset \mathbb{R}$.

Nesterov, Y. Squared functional systems and optimization problems, 2000.

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Polynomial interpolation

Fit a polynomial of degree n to a set of points (x_j, y_j),

$$f(x_j) = y_j, \quad j = 1, \dots, m,$$

i.e., linear equality constraints in c,

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

Semidefinite shape constraints

- Nonnegativity $f(x) \geq 0$.
- Monotonicity $f'(x) \geq 0$.
- Convexity $f''(x) \geq 0$.

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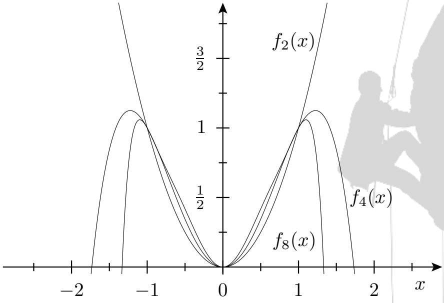
Smooth interpolation

Minimize largest derivative,

$$\begin{array}{ll} \text{minimize} & \max_{x\in[-1,1]} |f'(x)| \\ \text{subject to} & f(-1) = 1 \\ & f(0) = 0 \\ & f(1) = 1 \end{array}$$

or equivalently

$$\begin{array}{ll} \text{minimize} & z \\ \text{subject to} & -z \leq f'(x) \leq z \\ & f(-1) = 1 \\ & f(0) = 0 \\ & f(1) = 1. \end{array}$$



$$f_2(x) = x^2 \qquad f_4(x) = \frac{3}{2}x^2 - \frac{1}{2}x^4$$

$$f_2'(1) = 2 \qquad f_4'(\frac{1}{\sqrt{2}}) = \sqrt{2}$$

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