Exploiting special structure in semidefinite programs: an overview

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Standard form SDP

Primal problem

$$\min_{X \succeq 0} \operatorname{trace}(A_0 X) \quad \text{subject to} \quad \operatorname{trace}(A_k X) = b_k \quad (k = 1, \ldots, m),$$

where the data matrices $A_i \in \mathbb{S}^{n \times n}$ $(i = 0, \ldots, m)$ are linearly independent.

- $\mathbb{S}^{n \times n}$: symmetric $n \times n$ matrices;
- $X \succeq 0$: X symmetric positive semi-definite.

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Sometimes we will add the additional constraint $X\geq 0$ (componentwise nonnegative).

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3. if the A_i's lie in a low dimensional matrix algebra (this talk).

Matrix algebras

Definition

A set $\mathcal{A} \subseteq \mathbb{C}^{n\times n}$ (resp. $\mathbb{R}^{n\times n}$) is called a *matrix *-algebra* over $\mathbb{C}$ (resp. $\mathbb{R}$) if, for all $X, Y \in \mathcal{A}$:

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Assumption

There is a 'low dimensional' matrix *-algebra $\mathcal{A}_{SDP} \supseteq \{A_0, \ldots, A_m\}$.

Example

The circulant matrices form a commutative matrix *-algebra.

Form of a circulant matrix C

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & & c_{n-1} \\ c_{n-1} & c_0 & c_1 & & \cdots & \\ c_{n-2} & c_{n-1} & c_0 & c_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & c_1 \\ c_1 & & \cdots & & c_{n-1} & c_0 \end{bmatrix}.$$

Each row is a cyclic shift of the row above it, i.e: $C_{ij} = c_{i-j \bmod n}$ $(i, j = 0, \ldots, n-1)$.

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Consequence

We may restrict the primal problem to:

$$\min_{X \succeq 0} \left\{ \text{trace}(A_0 X) \ : \ \text{trace}(A_k X) = b_k \quad (k = 1, \ldots, m), \ X \in \mathcal{A}_{SDP} \right\}.$$

Canonical decomposition of a matrix *-algebra $\mathcal{A}$

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*Assume $\mathcal{A}$ is a matrix *-algebra over $\mathbb{C}$ that contains I.*

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$$Q^*\mathcal{A}Q = \begin{pmatrix} \mathcal{A}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{A}_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathcal{A}_s \end{pmatrix},$$

where each $\mathcal{A}_i \sim \mathbb{C}^{n_i \times n_i}$ for some integers n_i,

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where each $\mathcal{A}_i \sim \mathbb{C}^{n_i \times n_i}$ for some integers n_i, and takes the form

$$\mathcal{A}_i = \left\{ \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A \end{pmatrix} \middle| \; A \in \mathbb{C}^{n_i \times n_i} \right\} \quad (i = 1, \ldots, s).$$

Illustration: $\mathcal{A} \subset \mathbb{C}^{2438\times 2438}$

Sparsity pattern of a generic matrix in $\mathcal{A} \subset \mathbb{C}^{2438\times 2438}$:

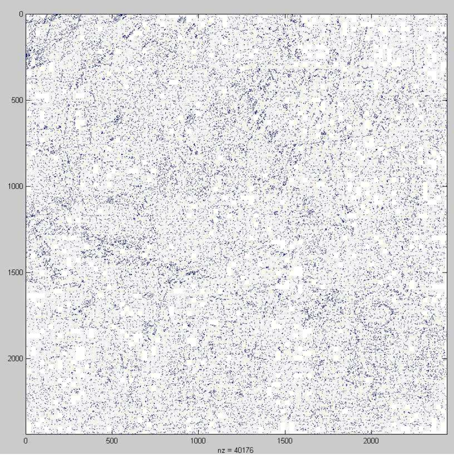
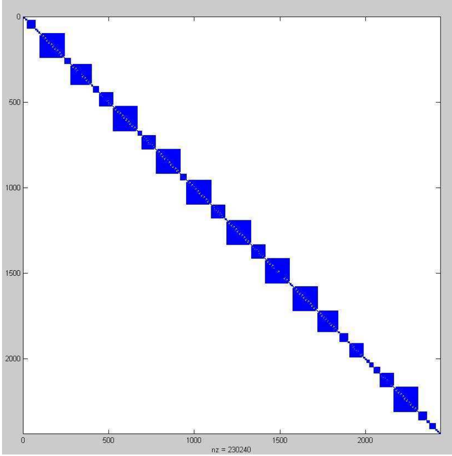


Illustration: $\mathcal{A} \subset \mathbb{C}^{2438\times 2438}$ (ctd.)

Simple components of $\mathcal{A}$ after unitary transformation:



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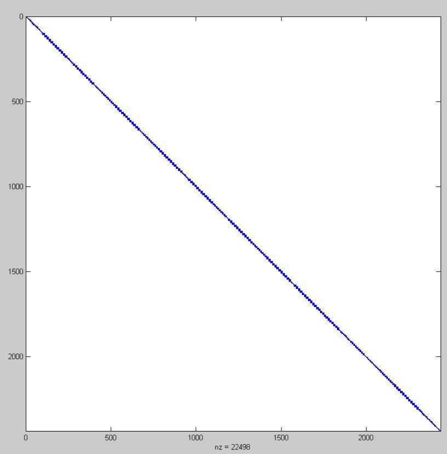
Irreducible components of $\mathcal{A}$ after second unitary transformation:



Example

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are diagonalized by the unitary (discrete Fourier transform) matrix:

$$Q_{ij} := \frac{1}{\sqrt{n}} e^{-2\pi\sqrt{-1}ij/n} \quad (i, j = 0, \ldots, n-1).$$

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If a basis of $\mathcal{A}$ is known, the unitary matrix Q may be computed using only linear algebra.

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Randomized algorithms:

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We give a simple prototype algorithm to illustrate the basic procedure ...

Simple block diagonalization algorithm

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The block sizes equal the multiplicities of the eigenvalues of A.

SDP reformulation

Assume we have a basis $B_1, \ldots, B_d$ of $\mathcal{A}_{SDP}$.

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Setting $X = \sum_{i=1}^{d} x_i B_i$, this becomes:

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- Delete any identical copies of blocks in the block structure.

Coherent configurations

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Consequence

If $\mathcal{A}_{SDP}$ is spanned by a coherent configuration and $X = \sum_{i=1}^d x_i B_i$, then

$$X \succeq 0 \text{ and } X \geq 0 \Longleftrightarrow \sum_{i=1}^d x_i B_i \succeq 0 \text{ and } x \geq 0.$$

Example

The circulant matrices have the basis

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & & \cdots & \\ 0 & 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & \\ 0 & & \cdots & & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & & \cdots & \\ & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & 1 \\ 1 & & \cdots & & 0 & 0 \end{bmatrix}, \ldots$$

and form an association scheme.

Stabilization

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We give an illustration of how the method works ...

Example: Weisfeiler-Leman Algorithm

What is the smallest coherent algebra that contains this matrix?

$$\begin{pmatrix} 0 & 3 & 2 & 2 & 4 & 4 \\ 3 & 0 & 4 & 4 & 2 & 2 \\ 2 & 4 & 1 & 4 & 4 & 4 \\ 2 & 4 & 4 & 1 & 4 & 4 \\ 4 & 2 & 4 & 4 & 1 & 4 \\ 4 & 2 & 4 & 4 & 4 & 1 \end{pmatrix}$$

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Replace equal entries by different, non-commuting variables:

$$\begin{pmatrix} 0 & 3 & 2 & 2 & 4 & 4 \\ 3 & 0 & 4 & 4 & 2 & 2 \\ 2 & 4 & 1 & 4 & 4 & 4 \\ 2 & 4 & 4 & 1 & 4 & 4 \\ 4 & 2 & 4 & 4 & 1 & 4 \\ 4 & 2 & 4 & 4 & 4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} t_0 & t_3 & t_2 & t_2 & t_4 & t_4 \\ t_3 & t_0 & t_4 & t_4 & t_2 & t_2 \\ t_2 & t_4 & t_1 & t_4 & t_4 & t_4 \\ t_2 & t_4 & t_4 & t_1 & t_4 & t_4 \\ t_4 & t_2 & t_4 & t_4 & t_1 & t_4 \\ t_4 & t_2 & t_4 & t_4 & t_4 & t_1 \end{pmatrix}.$$

Example: Weisfeiler-Leman Algorithm (ctd.)

Now square the symbolic matrix to get:

$$\begin{pmatrix} t_0 & t_3 & t_2 & t_2 & t_4 & t_4 \\ t_3 & t_0 & t_4 & t_4 & t_2 & t_2 \\ t_2 & t_4 & t_1 & t_4 & t_4 & t_4 \\ t_2 & t_4 & t_4 & t_1 & t_4 & t_4 \\ t_4 & t_2 & t_4 & t_4 & t_1 & t_4 \\ t_4 & t_2 & t_4 & t_4 & t_4 & t_1 \end{pmatrix}^2 = \begin{pmatrix} x_0 & x_2 & x_3 & x_3 & x_4 & x_4 \\ x_2 & x_0 & x_4 & x_4 & x_3 & x_3 \\ x_5 & x_6 & x_1 & x_7 & x_8 & x_8 \\ x_5 & x_6 & x_7 & x_1 & x_8 & x_8 \\ x_6 & x_5 & x_8 & x_8 & x_1 & x_7 \\ x_6 & x_5 & x_8 & x_8 & x_7 & x_1 \end{pmatrix},$$

where

$$\begin{aligned} x_0 &= t_0^2 + 2t_2^2 + t_3^2 + 2t_4^2 \\ x_1 &= t_1^2 + t_2^2 + 4t_3^2 \\ x_2 &= t_0 t_3 + 2t_2 t_4 + t_3 t_0 + 2t_4 t_2, \text{ etc.} \end{aligned}$$

Example: Weisfeiler-Leman Algorithm (ctd.)

Now square the symbolic matrix to get:

$$\begin{pmatrix} t_0 & t_3 & t_2 & t_2 & t_4 & t_4 \\ t_3 & t_0 & t_4 & t_4 & t_2 & t_2 \\ t_2 & t_4 & t_1 & t_4 & t_4 & t_4 \\ t_2 & t_4 & t_4 & t_1 & t_4 & t_4 \\ t_4 & t_2 & t_4 & t_4 & t_1 & t_4 \\ t_4 & t_2 & t_4 & t_4 & t_4 & t_1 \end{pmatrix}^2 = \begin{pmatrix} x_0 & x_2 & x_3 & x_3 & x_4 & x_4 \\ x_2 & x_0 & x_4 & x_4 & x_3 & x_3 \\ x_5 & x_6 & x_1 & x_7 & x_8 & x_8 \\ x_5 & x_6 & x_7 & x_1 & x_8 & x_8 \\ x_6 & x_5 & x_8 & x_8 & x_1 & x_7 \\ x_6 & x_5 & x_8 & x_8 & x_7 & x_1 \end{pmatrix},$$

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Repeat this process and add new symbols until convergence.

And, finally ...

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- ... but also pre-processing of some SDP's arising in optimal design (truss design, QAP, ...)
- More applications in polynomial optimization, graph coloring, ...

The End

Survey paper:

E. de Klerk. Exploiting special structure in semidefinite programming: A survey of theory and applications. *European Journal of Operational Research*, 201(1), 1–10, 2010.

Further reading:

F. Vallentin. Symmetry in semidefinite programs. *Linear Algebra and Appl.*, 430, 360–369, 2009.

THANK YOU!